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TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

Vol. III Part I.

I. *On the Spherical Aberration of the Eye-pieces of Telescopes.*

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[Read *May* 14 and *May* 21, 1827.]

In a paper on Achromatic Eye-pieces, which was read before this Society about three years since, and is printed in the second Volume of the Transactions, after giving the equations which must be satisfied in order that an eye-piece may be truly achromatic, I stated my intention of laying before the Society, at some future time, investigations of the conditions most favourable for the destruction of spherical aberration. I now proceed to fulfil this promise, by presenting to the Society an investigation, which I hope will be found pretty complete, of the course of a small pencil of homogeneous light after refraction by a lens; and by pointing out the application of the resulting expressions to the theory of eye-pieces. I have not considered the general case of a small pencil incident in any direction, but have confined myself to the cases in which the axis of the pencil passes through the axis of the lens: a limitation which, however, includes every thing relating to the eye-pieces of telescopes and microscopes.

The importance of these investigations has been acknowledged by every scientific artist, and by every writer who has endeavoured to assist, by practical rules, the common workman. But the complication of symbols has prevented most writers from entering upon

the subject at all: and those who have made some steps, have confined themselves to the simplest cases. I am not aware that any writer has investigated the conditions which must be satisfied that an object may be seen distinctly in all parts of the field of view, though this is the most important point in the construction of eye-pieces.

The effects of spherical aberration may be described as follows. The first is a distortion of the object. If, after examining an object in the center of the field of view, we bring it to the outside, we shall frequently find that it is extended in the direction of a radius of the field of view, and that it is increased, though in a smaller degree, in the other dimension. If we look at a square, it will appear to be drawn out at the corners, so that the sides are all convex towards the center. Sometimes the contrary effects will be produced: an object being less magnified at the circumference of the field than near the center. This defect may, in many cases, be entirely removed.

The second effect is, that if an object be distinctly visible in the center of the field of view, it is necessary to push in the eye-piece farther, in order to see clearly the objects at the outside of the field. In consequence of this, it is impossible, with the same position of the eye-piece, to see distinctly all parts of the field, or to see an object distinctly as it passes the field of view. This defect can never be destroyed.

The third effect is, that no adjustment of the place of the eye-piece will make an object distinctly visible when it is far from the center of the field of view. If a brilliant point, as a star, be viewed in this situation, with one position of the eye-piece, it appears a bright line in the direction of a radius of the field, and with another position it appears a bright line in a direction perpendicular to the former: with other positions it appears an ellipse, or a circle. In the case last mentioned, different parts of the last

image are formed at different distances from the eye: in this case, no distinct image is formed at all, except in the center of the field; the rays of other pencils never converging accurately to a point. This defect may frequently be entirely corrected, though it is sometimes prudent to leave it partially uncorrected.

But upon pursuing the investigations, it will appear that these conditions frequently interfere with one another, so that the construction, which is most advantageous for obviating one defect, makes another pretty large. And this has occasioned some anomalies in the practical rules which workmen have established. The lenses commonly in use are double equi-convex lenses, and plano-convex lenses: and the general rule of workmen is to place these in such a manner that the angle made by the incident ray with the first surface, shall be nearly equal to the angle made by the emergent ray with the second surface. This I believe is to diminish the distortion; and, for that purpose, the rule is not very far from the truth. But there is a remarkable departure from this law in the construction of the common eye-piece (Ramsden's) for transit instruments, and, generally, for all telescopes to which micrometers are applied. The eye-glass, nearest the object-glass, is a plano-convex, with its plane side towards the object-glass. This construction (which makes the angles above-mentioned extremely unequal) is adopted for the purpose, as workmen express it, of procuring a flat image; that is, for the purpose of making all parts of the image, after refraction, at the same distance from the eye. In this construction, then, the first condition is given up for the second: and the third is not at all considered, no general rule having been given for it.

It is necessary, then, to consider on which of these conditions the greatest stress should be laid. When the aperture of the telescope is very small, or its magnifying power very great, the breadth of a pencil is very small, and the second and third defects become

insensible. In this instance, then, it would be proper to attend to the first condition. But in all cases of micrometer measurement, in which the power is not very great, distortion is of little consequence, the object and the wires being equally distorted; but it is important that the object and the wires should be seen distinctly in all parts, without pushing or pulling the eye-piece. Here then the second condition should be insisted on, and the third, if possible, combined with it. For a telescope, which is used for examining the disc of a planet, or any similar observation without instrumental measurement, the third condition is quite as important as the second: the first less so. For common perspective glasses, opera glasses, &c. regard ought to be paid to all three, but especially to the first. I shall take occasion to apply the formulæ to the construction of the camera obscura: the second and third conditions ought there to be enforced, except the sacrifice of the first is very great.

The same causes which divide the effects of spherical aberration into three distinct parts, oblige us to use, for the investigation of these parts, three distinct operations. I shall, therefore, investigate, 1st, the course of the axis of a pencil; 2d, the point of convergence of rays in the plane passing through the axis of the pencil and the axis of the lens; 3d, the point of convergence of rays in the plane perpendicular to the former, and passing through the axis of the pencil.

It is evident that an object will be seen without distortion if its image, exactly similar to the object, be formed on a plane; and then the trigonometrical tangent of the angle, made with the axis of the lens by the axis of the pencil after refraction, will bear to the tangent of the angle before refraction a constant ratio: if the ratio be not constant, its difference from a constant ratio will indicate the degree of distortion. This suggests the following Propositions.

Prop. I. To find the proportion of the tangents of the angles made by the axis of a pencil with the axis of a lens before and after refraction.

Let AF, FG, GE, Fig. 1, be the course of the ray, and let FG produced meet the axis in D: draw FH, GK, perpendiculars to the axis: then our object is to compare the tangents of the angles FAH, GEK; or to find the value of $\frac{GK}{FH} \cdot \frac{AH}{EK}$. Let $FH = a$, $BC = t$, $AB = b$, $CE = c$, $BD = x$; and let B, C, X, be the values of the latter quantities when $a = 0$. Draw FL perpendicular to KG produced. Then

$$GK = a - GL = a - \frac{a \,.\, HK}{HD} = a - a \,.\, \frac{t - BH - KC}{X},$$

omitting small quantities in the estimation of HD;

$$\therefore \frac{GK}{FH} = 1 - \frac{t}{X} + \frac{BH + KC}{X}.$$

Now I will remark, that we may at once omit t in this expression, though it is larger than the next term which is to be preserved. The reason is that $\frac{t}{X}$ is independent of the aperture, and as it is our object to examine, among the small parts, only those which depend on the aperture, there is no necessity for preserving a small constant term. The product of t and small variable quantities being extremely small, is, of course, to be neglected. It appears, then, that though in the beginning of this and other investigations, we shall be obliged to take t into account, we may always expunge it in a subsequent stage. When multiplied by a constant, the product is to be rejected because small and independent of the aperture: when multiplied by a variable, which is necessarily small, the product will be of an order smaller than the quantities taken into account. Hence,

$$\frac{GK}{FH} = 1 + \frac{BH + KC}{X}.$$

Let the radius of the first surface be r, that of the second s. Then

$$BH = \frac{a^2}{2r},\ KC = \frac{a^2}{2s} \text{ nearly; } \therefore \frac{GK}{FH} = 1 + \frac{1}{X}\left(\frac{1}{r} + \frac{1}{s}\right) \cdot \frac{a^2}{2}.$$

$$\text{Also } \frac{AH}{EK} = \frac{b + \frac{a^2}{2r}}{c + \frac{a^2}{2s}} = \frac{b}{c}\left(1 + \frac{a^2}{2br} - \frac{a^2}{2cs}\right), \text{ neglecting } a^4;$$

$$\therefore \frac{GK}{FH} \cdot \frac{AH}{EK} = \frac{b}{c}\left\{1 + \left(\frac{1}{r}\left(\frac{1}{X} + \frac{1}{b}\right) - \frac{1}{s}\left(\frac{1}{c} - \frac{1}{X}\right)\right)\frac{a^2}{2}\right\}.$$

Now, by the common approximate formulæ for the principal foci at refraction by spherical surfaces,

$$\frac{1}{X} = \frac{n-1}{nr} - \frac{1}{nB}, \text{ and } \frac{1}{X} = \frac{1}{nC} - \frac{n-1}{ns}:$$

$$\text{hence, } \frac{1}{X} + \frac{1}{b}, \text{ or } \frac{1}{X} + \frac{1}{B} = \frac{n-1}{n}\left(\frac{1}{r} - \frac{1}{B}\right),$$

$$\text{and } \frac{1}{c} - \frac{1}{X} = \frac{n-1}{n}\left(\frac{1}{C} + \frac{1}{s}\right);$$

$$\therefore \frac{GK}{FH} \cdot \frac{AH}{EK} = \frac{b}{c}\left\{1 + \frac{n-1}{n}\left(\frac{1}{r} \cdot \overline{\frac{1}{r} - \frac{1}{B}} - \frac{1}{s} \cdot \overline{\frac{1}{s} + \frac{1}{C}}\right)\frac{a^2}{2}\right\}.$$

This expression may be put into a more convenient form thus. Let the principal focal length of the lens $= F$; let $\frac{1}{F} = f$. Then, neglecting the thickness,

$$\frac{1}{B} + \frac{1}{C} = f; \quad \frac{1}{r} + \frac{1}{s} = \frac{f}{n-1}.$$

$$\text{Let therefore } \frac{1}{r} = \frac{f}{2(n-1)} + \varepsilon; \ \therefore \frac{1}{s} = \frac{f}{2(n-1)} - \varepsilon:$$

$$\text{and let } \frac{1}{B} = \frac{f}{2} + e; \ \therefore \frac{1}{C} = \frac{f}{2} - e.$$

Substituting, the proportion of the tangents becomes

$$\frac{b}{c}\left\{1+\frac{f}{n}\left(\overline{n+1}\,.\,v+e\right).\frac{a^2}{2}\right\}=\frac{B+\overline{b-B}}{C+\overline{c-C}}\left\{1+\frac{f}{n}\left(\overline{n+1}\,.\,v+e\right)\frac{a^2}{2}\right\}$$

$$=\frac{B}{C}\left\{1+\frac{b-B}{B}-\frac{c-C}{C}+\frac{f}{n}\left(\overline{n+1}\,.\,v+e\right).\frac{a^2}{2}\right\}.$$

The quantity $c-C$ may be thus calculated. By the known formulæ for aberration at refraction by one surface (*Cambridge Trans.* Vol. II. p. 110),

$$\frac{1}{x}=\frac{n-1}{n}\cdot\frac{1}{r}-\frac{1}{nb}+\frac{a^2}{2}\cdot\frac{n-1}{n^3}\left(\frac{1}{r}+\frac{1}{B}\right)^2\cdot\left(\frac{1}{r}+\frac{n+1}{B}\right).$$

Rejecting the thickness, the same formula, putting $-s$ for r, $-x$ for b, c for x, and $\frac{1}{n}$ for n, gives

$$\frac{1}{c}=\overline{n-1}\,\frac{1}{s}+\frac{n}{x}+\frac{a^2}{2}\,.\,n^2\,.\,\overline{n-1}\,.\left(\frac{1}{s}+\frac{1}{X}\right)^2\cdot\left(\frac{1}{s}+\frac{n+1}{nX}\right).$$

Substituting for $\frac{1}{x}$ the value just found, and for $\frac{1}{X}$ in the small term the value before set down, we find

$$\frac{1}{c}=\overline{n-1}\left(\frac{1}{s}+\frac{1}{r}\right)-\frac{1}{b}$$

$$+\frac{a^2}{2}\cdot\frac{n-1}{n^2}\left\{\left(\frac{1}{r}+\frac{1}{B}\right)^2\cdot\overline{\frac{1}{r}+\frac{n+1}{B}}+\left|\overline{\frac{1}{s}+\frac{1}{C}}\right|^2\cdot\overline{\frac{1}{s}+\frac{n+1}{C}}\right\}.$$

Substituting for $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{B}$, and $\frac{1}{C}$ the values found above, and putting $b=B+\overline{b-B}$, we have

$$\frac{1}{c}=\overline{n-1}\left(\frac{1}{s}+\frac{1}{r}\right)-\frac{1}{B}+\frac{b-B}{B^2}$$

$$+\frac{a^2}{2}\left\{\frac{n^2f^3}{4\,(n-1)^2}+\frac{f}{n}\,.\,\overline{v+e}\,\left(\overline{n+2}\,v+\overline{3\,n+2}\,e\right)\right\}$$

$$=\frac{1}{C}+\frac{b-B}{B^2}+\frac{a^2}{2}\left\{\frac{n^2f^3}{4\,(n-1)^4}+\frac{f}{n}\,.\,\overline{v+e}\,.\left(\overline{n+2}\,.\,v+\overline{3\,n+2}\,e\right)\right\}$$

$\Big($since the part independent of a must be the value of $\frac{1}{c}$ when $a=0$, that is, must $=\frac{1}{C}\Big)$; and taking the reciprocal, and subtracting C,

$$c-C=-\frac{C^2}{B^2}(b-B)$$
$$-\frac{a^2}{2}.C^2.\left\{\frac{f}{n}.(\overline{n+2}.v^2+\overline{4n+4}\,ev+\overline{3n+2}\,e^2)+\frac{n^2f^3}{4(n-1)^2}\right\}.$$

Substituting this value of $c-C$ in the expression for the proportion of the tangents, it becomes

$$\frac{B}{C}\left\{1+\frac{B+C}{B^2}(b-B)+\frac{a^2}{2}\left(\frac{f}{n}(\overline{n+1}\,v+e)\right.\right.$$
$$\left.\left.+\frac{Cf}{n}(\overline{n+2}.v^2+\overline{4n+4}.ev+\overline{3n+2}\,e^2)+\frac{n^2Cf^3}{4(n-1)^2}\right)\right\}.$$

This, for abbreviation, we shall call $\frac{B}{C}\{1+Q\}$. The first term of Q may sometimes be more conveniently calculated by observing that

$$\frac{1}{F}=\frac{1}{B}+\frac{1}{C}=\frac{B+C}{BC},\text{ and therefore }\frac{B+C}{B^2}=\frac{C}{BF}=\frac{C}{B}f.$$

PROP. II. To find the proportion of the tangents of the angles after refraction through several lenses.

The proportion of the tangent of the first angle to the second being $\frac{B}{C}\{1+Q\}$, that of the second to the third $\frac{B'}{C'}\{1+Q'\}$, &c., the proportion of the first to the last will be $\frac{BB', \&c.}{CC', \&c.}.\{1+Q+Q'+\&c.\}$: and it is only necessary to show how Q, Q', &c. are to be calculated. First, it will easily be seen that $a'=\frac{B'}{C}a$, $a''=\frac{B''}{C'}a'$, &c. Then, since $c+b'=$interval between first and second lenses $=C+B'$, we have

$$b'-B'=-(c-C),$$

and, therefore, $b' - B'$ can be calculated by the expression

$$\frac{C^2}{B^2}(b-B) + \frac{a^2}{2}.\left\{\frac{C^2 f}{n}.(\overline{n+2}.v^2 + \overline{4n+4}\,ev + \overline{3n+2}\,e^2 + \frac{n^2 C^2 f^3}{4(n-1)^2}\right\}.$$

Finally, by substitution in the expression above, the values of Q, Q', &c. can be found.

These expressions fail when $B = F$. As the lens in this case is always followed by another, we may investigate the effect of the combination. In Fig. 2, let A be near the principal focus of the first lens. The ray, after refraction, meets the axis at a very distant point E. Now we have found that

$$\frac{1}{c} = \frac{1}{F} - \frac{1}{b} + P\frac{a^2}{2};$$

$$\text{where } P = \frac{f}{n}(\overline{n+2}.v^2 + \overline{4n+4}.ev + \overline{3n+2}.e^2) + \frac{n^2 f^3}{4(n-1)^2},$$

or, in this instance,

$$= f\left(\frac{n+2}{n}v^2 + \frac{2n+2}{n}fv + \frac{3n+2}{4n}f^2 + \frac{n^2}{4(n-1)^2}f^2\right);$$

$$\text{or } \frac{1}{c} = \frac{1}{F} - \frac{1}{B} + \frac{b-B}{B^2} + P\frac{a^2}{2} = \frac{b-B}{B^2} + P.\frac{a^2}{2}:$$

c, therefore, is very large. Let I be the distance between the lenses. The ray falls on the second lens tending to converge to the distance $c - I$ beyond it, and, therefore, after refraction, it meets the axis at the distance c', where

$$\frac{1}{c'} = \frac{1}{c-I} + \frac{1}{F'} + P'.\frac{a^2}{2},$$

$$P' \text{ being } = f'\left(\frac{n+2}{n}v'^2 - \frac{2n+2}{n}f'v' + \frac{3n+2}{4n}f'^2 + \frac{n^2}{4(n-1)^2}f'^2\right).$$

Now $\frac{1}{c-I} = \frac{1}{c} + \frac{I}{c^2} + \text{\&c.}$, and $\frac{I}{c^2}$ is of the order a^4, and, therefore, to be neglected. Hence,

$$\frac{1}{c'} = \frac{1}{F'} + \frac{b-B}{B^2} + P\frac{a^2}{2} + P'\frac{a^2}{2}:$$

whence (since $B = F$),

$$c' = F' - \frac{F'^2}{F^2}(b-B) - \frac{a^2}{2} . F'^2 (P+P').$$

Hence, $b'' - B''$ will $= \frac{F'^2}{F^2}(b-B) + \frac{a^2}{2} . F'^2 (P+P')$.

And $E'K' = c' + \frac{a^2}{2s'} = F' - \frac{F'^2}{F^2}(b-B) - \frac{a^2}{2} . F'^2 (P+P') + \frac{a^2}{2} . \frac{1}{s'}$.

Again, $F'H' = GK - \frac{I}{c}a =$ (putting for GK the value found before)

$$a + \frac{1}{X}\left(\frac{1}{r} + \frac{1}{s}\right) . \frac{a^3}{2} - \frac{I}{F^2}(b-B) . a - IP\frac{a^3}{2}.$$

Hence, (as before)

$$G'K' = F'H' + \frac{1}{X'}\left(\frac{1}{r'} + \frac{1}{s'}\right) . \frac{a^3}{2}$$

$$= a + \left\{\frac{1}{X}\left(\frac{1}{r} + \frac{1}{s}\right) + \frac{1}{X'}\left(\frac{1}{r'} + \frac{1}{s'}\right) - P\right\}\frac{a^3}{2} - \frac{I}{F^2}(b-B)\, a.$$

Now, the tangent of the angle made with the axis after refraction at the second lens, is $\frac{G'K'}{E'K'}$: that before incidence, is

$$\frac{FH}{AH} = \frac{a}{F + \overline{b-B} + \frac{a^2}{2r}}.$$

Dividing the former by the latter, the proportion required, after all substitutions, is $\frac{F}{F'}(1 + Q_1)$, where

$$Q_1 = \frac{a^2}{2}\left\{\frac{f}{n}\left(\overline{n+1}\,v + \frac{f}{2}\right) + \frac{f'}{n}\left(\overline{n+1} . v' - \frac{f'}{2}\right) - (I - F')\,P + F'P'\right\}$$

$$- \frac{I - F - F'}{F^2}(b-B).$$

Ex. 1. In the common opera-glass, magnifying four times, to find the distortion.

As the eye is supposed to be in contact with the eye-glass, the axis of every effective pencil must pass through the center of the eye-glass. In this case, then, we must trace the ray backwards, and examine its course after passing the object-glass:

$$B \text{ will } = \frac{3F}{4},\quad C = -3F,\quad \frac{1}{B} = \frac{4f}{3},\quad e = \frac{1}{B} - \frac{f}{2} = \frac{5f}{6}:\ b - B = 0:$$

whence
$$Q = \frac{a^2}{2}\left\{\frac{f}{n}\left(\overline{n+1}\,.\,v + \frac{5f}{6}\right) - \frac{3}{n}\left(\overline{n+2}\,.\,v^2 + \overline{4n+4}\,.\,\frac{5f}{6}v\right.\right.$$
$$\left.\left. + \overline{3n+2}\,.\,\frac{25}{36}f^2\right) - \frac{3n^2f^2}{4(n-1)^2}\right\}.$$

This may be put under the form

$$-\frac{3(n+2)}{n}\left\{v + \frac{3(n+1)}{2(n+2)}f\right\}^2 - \left\{\frac{3n^2}{4(n-1)^2} - \frac{6n^2-28n+1}{12n\,.\,\overline{n+2}}\right\}f^2:$$

or, supposing $n = 1{,}5$ (which is sufficiently exact),

$$Q = -\frac{a^2}{2}\left\{7\left(v + \frac{15}{14}f\right)^2 + \frac{1811}{252}f^2\right\}.$$

Hence, we find that

1st. Q cannot be made $= 0$.

2d. Q is smallest when $v = -\frac{15}{14}f$: which gives $r = -14\,.\,F$, (the negative sign showing that the surface is concave)

$$s = \frac{14}{29}F,\quad Q = -\frac{1811}{504}\cdot\frac{a^2}{F^2}.$$

The object-glass, therefore, is slightly meniscus; the exterior side being convex.

3d. If the lens be plano-convex, the plane side towards the eye-glass,

$$v = -f,\ \text{and}\ Q = -\frac{1820}{504}\cdot\frac{a^2}{F^2}.$$

4th. If the lens be equi-convex, $v=0$, and

$$Q=-\frac{3836}{504}\cdot\frac{a^2}{F^2}.$$

5th. If the lens be plano-convex, the convexity towards the eye-glass, $v=f$, and

$$Q=-\frac{9380}{504}\cdot\frac{a^2}{F^2}.$$

Of the common forms, then, the plano-convex, its plane side towards the eye-glass, is much the best.

Ex. 2. The axis of the pencils of rays diverge from a very distant point (a supposition that we shall make in all the succeeding examples) and fall on a convex lens. (This is the case of a single eye-glass applied to the astronomical telescope.)

In this and subsequent examples, we trace the ray in the direction in which it really proceeds.

Here $b-B=0$; and $C=F$, $e=-\frac{f}{2}$:

$$\text{whence } Q=\frac{a^2}{2}\left\{\frac{f}{n}\left(\overline{n+1}\,.v-\frac{f}{2}\right)\right.$$

$$\left.+\frac{1}{n}\left(\overline{n+2}\,.v^2-\overline{3n+2}\,.fv+\frac{3n+2}{4}f^2\right)+\frac{n^2f^2}{4(n-1)^2}\right\}$$

$$=\frac{a^2}{2}\left\{\frac{7}{3}.\overline{v-\frac{5}{14}f}\,\Big|^2+\frac{227}{84}f^2\right\}. \quad \text{Hence,}$$

1st. Q cannot be made $=0$.

2d. Q is a minimum when $v=\frac{5}{14}f$: which gives

$$r=\frac{14}{19}F,\quad s=\frac{14}{9}F,\quad Q=\frac{227}{168}\cdot\frac{a^2}{F^2}.$$

3d. If the lens be plano-convex, its plane side towards the object-glass, $v=-f$, and

$$Q=\frac{588}{168}\cdot\frac{a^2}{F^2}.$$

4th. If the lens be equi-convex, $v=0$, and

$$Q=\frac{252}{168}\cdot\frac{a^2}{F^2}.$$

5th. If the lens be plano-convex, its convex side towards the object-glass, $v=f$, and

$$Q=\frac{308}{168}\cdot\frac{a^2}{F^2}.$$

Of the common forms, then, the equi-convex is the best: it would be somewhat improved by increasing the curvature of the surface next the object-glass.

Ex. 3. Suppose two convex lenses of equal focal length are placed in contact.

If we suppose $n=1,5$, the equations of Prop. 1, and 2, may be put under this form:

$$Q=\frac{C}{BF}(b-B)+a^2\left\{\frac{5}{6}fv+\frac{1}{3}fe+Cf\left(\frac{7}{6}v^2+\frac{10}{3}ev+\frac{13}{6}e^2+\frac{9}{8}f^2\right)\right\};$$

$$b'-B'=\frac{C^2}{B^2}(b-B)+a^2.C^2f\left(\frac{7}{6}v^2+\frac{10}{3}ev+\frac{13}{6}e^2+\frac{9}{8}f^2\right);$$

$$a'=\frac{B'}{C}a.$$

In the present instance,

$$C=F,\ B'=-F,\ C'=\frac{F}{2},\ e=-\frac{f}{2},\ e'=-\frac{3f}{2},\ b-B=0,\ a'=a.$$

First, then, we find

$$b'-B'=a^2.\frac{1}{f}\left\{\frac{7}{6}v^2-\frac{5}{3}fv+\frac{5}{3}f^2\right\}:$$

$$\text{then } Q = a^2 \left\{\frac{5}{6} fv - \frac{1}{6} f^2 + \left(\frac{7}{6} v^2 - \frac{5}{3} fv + \frac{5}{3} f^2\right)\right\}:$$

and (by using the value of $b' - B'$ just found),

$$Q' = -a^2 \cdot \frac{1}{2} \left\{\frac{7}{6} v^2 - \frac{5}{3} fv + \frac{5}{3} f^2\right\}$$

$$+ a^2 \left\{\frac{7}{12} v'^2 - \frac{5}{3} fv' + \frac{5}{2} f^2\right\}.$$

Adding together Q and Q', and putting R and R' for those parts which depend respectively on the first and second lens,

$$R = a^2 \left(\frac{7}{12} v^2 + \frac{2}{3} f^2\right), \quad R' = a^2 \left(\frac{7}{12} \cdot \overline{v' - \frac{10}{7} f}\Big|^2 + \frac{110}{84} f^2\right).$$

1st. R is least when $v = 0$, or the first lens is equi-convex: then, its value $= \frac{2}{3} \cdot \frac{a^2}{F^2}$. If it be plano-convex, in either position,

$$R = \frac{5}{4} \cdot \frac{a^2}{F^2}.$$

2d. R' is least when $v' = \frac{10}{7} f$, which gives

$$r' = \frac{7}{17} F, \quad s' = -\frac{7}{3} F, \quad R' = \frac{110}{84} \cdot \frac{a^2}{F^2}.$$

If the second lens be plano-convex, its plane side toward the first lens, $R' = \frac{399}{84} \cdot \frac{a^2}{F^2}$. If equi-convex, $R' = \frac{210}{84} \cdot \frac{a^2}{F^2}$. If plano-convex in the other position, $R' = \frac{119}{84} \cdot \frac{a^2}{F^2}$. Of the common forms, then, the plano-convex, its convexity towards the first lens, is best; it ought to be slightly meniscus.

3. A single lens to produce the same effect, must have had a focal length $\frac{F}{2}$, and, therefore, by the last Example, the smallest

value of Q would have been $\frac{454}{84} \cdot \frac{a^2}{F^2}$. In the combination of two lenses, the smallest value of $Q + Q' = R + R'$ is

$$\frac{2}{3} \cdot \frac{a^2}{F^2} + \frac{110}{84} \cdot \frac{a^2}{F^2} = \frac{166}{84} \cdot \frac{a^2}{F^2}.$$

The distortion, then, is little more than one-third of what it would be with a single eye-glass.

Ex. 4. Suppose the eye-piece to be the Huyghenian eye-piece; the first and second lenses having respectively the focal lengths $3M$ and M, and their distance being $2M$.

Let $\frac{1}{M} = m$. Here $C \doteq F = 3M$; $B' = -M$, $C' = \frac{M}{2}$,

$$e = -\frac{f}{2} = -\frac{m}{6}, \quad e' = -\frac{3f'}{2} = -\frac{3m}{2}, \quad a' = \frac{a}{3}.$$

By a process of the same kind we find

$$R = a^2 \left\{ \frac{4}{189} m^2 - \frac{7}{12} \cdot \overline{v - \frac{10}{21} m}\Big|^2 \right\},$$

$$R' = a^2 \left\{ \frac{55}{378} m^2 + \frac{7}{108} \cdot \overline{v' - \frac{10}{7} m}\Big|^2 \right\},$$

$$\text{and } R + R' = a^2 \left\{ \frac{m^2}{6} + \frac{7}{108} \cdot \overline{v' - \frac{10}{7} m}\Big|^2 - \frac{7}{12} \cdot \overline{v - \frac{10}{21} m}\Big|^2 \right\}.$$

Hence, we find

1st. $R + R'$ or $Q + Q'$ may be made $= 0$, by assuming any value for v', and determining v. This eye-piece, then, may be made absolutely free from distortion.

2d. If the first lens be plano-convex, the convex side toward the object-glass,

$$v = f = \frac{m}{3}, \text{ and } R = \frac{a^2}{108 M^2}.$$

If equi-convex, $R = -\frac{12\,a^2}{108\,M^2}$. If plano-convex, in the other position,

$$v = -\frac{m}{3}, \text{ and } R = -\frac{39\,a^2}{108\,M^2}.$$

3d. If the second lens be plano-convex, the convex side toward the first lens, $v' = m$, and $R' = \frac{17}{108} \cdot \frac{a^2}{M^2}$. If equi-convex,

$$R' = \frac{30}{108} \cdot \frac{a^2}{M^2}.$$

If plano-convex in the other position,

$$v' = -m, \text{ and } R' = \frac{57}{108} \cdot \frac{a^2}{M^2}.$$

4th. The most favourable combination of common lenses is the first equi-convex, and the second plano-convex, with its convex side toward the first lens. This gives

$$R + R' = \frac{5}{108} \cdot \frac{a^2}{M^2}.$$

The next best is, the first plano-convex, its plane side toward the object-glass, and the second equi-convex, which gives

$$R + R' = -\frac{9}{108} \cdot \frac{a^2}{M^2}.$$

Next are, either two plano-convex lenses with their convexities turned the same way (it is indifferent which way), or two equi-convex lenses, all which give

$$R + R' = \frac{18}{108} \cdot \frac{a^2}{M^2}.$$

5th. When the second lens is plano-convex, its convex side toward the first lens, the distortion will be destroyed if

$$r = \frac{14}{19} M, \quad s = -\frac{42}{29} M, \quad \text{or if } r = \frac{42}{11} M, \quad s = \frac{42}{17} M;$$

that is, if the first lens be a meniscus, its convexity toward the object-glass, and the radius of its concavity about double that of its convexity, or if the first lens be double-convex, the convexity being somewhat greatest on the side next the second lens.

6th. A single lens to produce the same effect must have a focal length $\frac{3M}{2}$, and, therefore, its smallest value of Q would be

$$\frac{65}{108} \cdot \frac{a^2}{M^2}.$$

Ex. 5. The eye-piece is the old erecting eye-piece, consisting of three lenses of equal focal length M, placed at intervals of $2M$.

It will easily be seen that $B' = F'$, and, therefore, we must recur to the investigation for that particular case. Now, as $F = F' = M$, and $I = 2M$, $I - F - F'$ will $= 0$. To calculate $b - B$ is, therefore, unnecessary: we have merely to add the expressions for Q and Q_1: and $a = a' = a''$. Selecting at once the parts belonging to each lens,

$$R = \frac{a^2}{2}\left\{\frac{f}{n}\left(\overline{n+1}\,.v - \frac{f}{2}\right) + \frac{n+2}{n}v^2 - \frac{2n+2}{n}fv + \frac{3n+2}{4n}f^2 + \frac{n^2 f^2}{4(n-1)^2}\right\}:$$

$$R' = \frac{a^2}{2}\left\{\frac{f'}{n}\left(\overline{n+1}\,.v' + \frac{f'}{2}\right) - (I - F'')f'\left(\frac{n+2}{n}v'^2 + \frac{2n+2}{n}f'v'\right.\right.$$

$$\left.\left. + \frac{3n+2}{4n}f'^2 + \frac{n^2}{4(n-1)^2}f'^2\right)\right\}:$$

$$R'' = \frac{a^2}{2}\left\{\frac{f''}{n}\left(\overline{n+1}\,.v'' - \frac{f''}{2}\right) + F''f''\left(\frac{n+2}{2}v''^2 - \frac{2n+2}{n}f''v''\right.\right.$$

$$\left.\left. + \frac{3n+2}{4n}f''^2 + \frac{n^2}{4(n-1)^2}f''^2\right)\right\}.$$

Making the substitutions,

$$R = a^2\left(\overline{\frac{7}{6}.v - \frac{5}{14}m}\,\Big|^2 + \frac{227}{168}m^2\right),$$

$$R' = -a^2\left(\overline{\frac{7}{6}.v' + \frac{5}{14}m}\Big|^2 + \frac{227}{168}m^2\right),$$

$$R'' = a^2\left(\overline{\frac{7}{6}.v'' - \frac{5}{14}m}\Big|^2 + \frac{227}{168}m^2\right).$$

1st. $R + R' + R''$ may be made $= 0$, or the distortion may be entirely destroyed.

2d. If the first lens be plano-convex, its plane side toward the object-glass, $R = \frac{21}{6} . \frac{a^2}{M^2}$. If equi-convex, $R = \frac{9}{6} . \frac{a^2}{M^2}$. If plano-convex, with its convexity toward the object-glass, $R = \frac{11}{6} . \frac{a^2}{M^2}$.

3d. The same values apply, without alteration, to the third lens. For the second, their order must be reversed, and they must be made negative.

4th. The best combination of common lenses is, all plano-convex, with their convexity toward the object-glass: then

$$R + R' + R'' = \frac{a^2}{6M^2}.$$

If we make either the first or the third equi-convex, then

$$R + R' + R'' = -\frac{a^2}{6M^2}.$$

If both be equi-convex,

$$R + R' + R'' = -\frac{a^2}{2M^2}.$$

5th. A single lens to produce the same power must have a focal length M, and, therefore, its smallest value of Q is $\frac{227}{168} . \frac{a^2}{M^2}$.

Ex. 6. The eye-piece consists of four lenses of focal lengths $3\,M$, $4\,M$, $4\,M$, and $3\,M$, placed at the intervals $4\,M$, $6\,M$, and $5{,}13\,M$. (These numbers satisfy the equation of achromatism, ***Cambridge Transactions***, Vol. II, p. 247, and give a convenient form of the four-glass eye-piece.)

$$\text{Here } C = 3\,M,\ B' = M,\ C' = -\frac{4}{3}\,M,\ B'' = \frac{22}{3}\,M,$$

$$C'' = \frac{44}{5}\,M,\ B''' = -3{,}67\,.\,M,\ C''' = 1{,}65\,M.$$

$$\text{Hence, } e = -\frac{m}{6},\ e' = \frac{7\,m}{8},\ e'' = \frac{m}{88},\ e''' = -m \times {,}439.$$

$$\text{Also } a' = \frac{a}{3},\ \ a'' = \frac{11}{6}\,a,\ \ a''' = {,}7645 \times a.$$

From these data we find, in the same way,

$$\begin{aligned}
R &= \ \ a^2\,\{\,{,}5227\,(v\ + {,}0276\,m)^2 + {,}0640\,m^2\},\\
R' &= -a^2\,\{\,{,}0384\,(v' +\ {,}948\,m)^2 + {,}0142\,m^2\},\\
R'' &= -a^2\,\{2{,}7606\,(v'' - {,}1106\,m)^2 + {,}1300\,m^2\},\\
R''' &= \ \ a^2\,\{\,{,}3750\,(v''' - {,}4107\,m)^2 + {,}0827\,m^2\}.
\end{aligned}$$

From which we derive the following conclusions.

1st. $Q + Q' + Q'' + Q'''$, or $R + R' + R'' + R'''$, may always be made $= 0$, with any assumed values of v and v'''. And, in most cases, we may assume any three of the quantities, v, v', v'', v''', and it will be possible to find the fourth, so that $R + R' + R'' + R''' = 0$.

2d. If the first lens be plano-convex, its convex side towards the object-glass, $v = f = \frac{m}{3}$, and $R = \frac{a^2}{M^2}\,.\,{,}1322$. If equi-convex, $v = 0$, and $R = \frac{a^2}{M^2}\,.\,{,}0644$. If plano-convex, its plane side toward the object-glass, $v = -\frac{m}{3}$, and $R = \frac{a^2}{M^2}\,.\,{,}1129$.

3d. If the second lens be plano-convex, its convex side towards the first, $R' = -\frac{a^2}{M^2} \cdot ,0694$. If equi-convex, $R' = -\frac{a^2}{M^2} \cdot ,0488$. If plano-convex, its plane side towards the first lens,

$$R' = -\frac{a^2}{M^2} ,0329.$$

4th. If the third lens be plano-convex, its convex side towards the second, $R'' = -\frac{a^2}{M^2} \cdot ,1836$. If equi-convex, $R'' = -\frac{a^2}{M^2} \cdot ,1639$. If plano-convex, its plane side towards the second lens,

$$R'' = -\frac{a^2}{M^2} \cdot ,4889.$$

5th. If the fourth lens be plano-convex, its convex side towards the third, $R''' = \frac{a^2}{M^2} \cdot ,0849$. If equi-convex, $R''' = \frac{a^2}{M^2} \cdot ,1459$. If plano-convex in the other position, $R''' = \frac{a^2}{M^2} \cdot ,2885$.

6th. The most favourable combination of these is, if all the lenses be plano-convex, the first, third, and fourth, having their convexities towards the object-glass, and the second its plane side turned the same way; they give

$$R + R' + R'' + R''' = \frac{a^2}{M^2} \cdot ,0006.$$

The next is, if the first and second be plano-convex, with their plane sides towards the object-glass, the third equi-convex, and the fourth plano-convex, with its convexity towards the object-glass: they give

$$R + R' + R'' + R''' = \frac{a^2}{M^2} ,0010.$$

The next is, if all the lenses be equi-convex, which makes

$$R + R' + R'' + R''' = -\frac{a^2}{M^2} \cdot ,0024.$$

7th. A single eye-glass to produce the same effect must have a focal length $\frac{CC'C''C'''}{B'B''B'''} = M \times 2{,}16$, and, therefore, its smallest value of Q would be $\frac{a^2}{M^2} \times {,}2896$.

These instances will probably be sufficient, as exemplifying the application of the formulæ and as giving results of some practical value. It appears at first somewhat curious that (as in Ex. 3.) the spherical aberration of one lens should correct that of another, when the ray is received on the second before its intersection with the axis of the lens. This may be explained nearly in the same way as the correction of chromatic aberration by the same construction (*Cambridge Transactions*, Vol. II. p. 231.). The ray converging to a point nearer than that to which, without spherical aberration, it would converge, is incident on the second lens at a point where the refracting angle is so small that it emerges in a direction parallel to that which it would have had if there had been no aberration. Of the mutual correction of aberration in the more complicated eye-pieces, a similar explanation may be given.

Having considered the course of the axis of a pencil of rays, I shall now consider the convergence of the rays in each pencil. In general, the rays of a small pencil diverging from a point, after refraction by a lens do not converge to a point, but pass through two straight lines in different planes. To elucidate this, suppose in Fig. 3, one ray AF of a pencil, diverging from a point in the axis of the lens BC, to be refracted in the direction GE; then another ray at a very small distance from AF, in the plane perpendicular to the paper, and passing through AF (which we shall call the *perpendicular plane*), will also converge to E; that is, two rays in the perpendicular plane meet at E. In the same manner, it

will be seen that all the rays of the pencil after refraction pass through different parts of the line Ee. Now, let Af be another ray near AF, and also in the plane of the paper (which we shall call the *paper plane*), and let it be refracted in the direction ge. It is evident that GE and ge converge to the point M: and it will easily be seen that all rays in the paper plane, or in a plane parallel to the paper plane (the breadth of the pencil being small) converge to a point in a line drawn through M perpendicular to the plane of the paper. That is, all the rays of the pencil pass through a line perpendicular to the paper at the point M, and all pass through the line Ee in the plane of the paper. And it is plain that all the rays of the pencil do not converge to any one point whatever.

In the same manner it may be understood, that when a pencil of rays diverges from a point L, Fig. 4, which is not in the axis of a lens, all the rays after refraction may pass through one straight line perpendicular to the paper at some point S, and through another straight line in the paper plane, as Mm. That this is the case, our investigation will shew. The mode, therefore, of discovering the course of a pencil of rays after refraction is to examine the position of the two lines through which, after refraction, all the rays of the pencil must pass.

I shall first remark, that in Fig. 3, we might, without sensible error, have asserted that all the rays of the pencil pass through the line Ek (which is perpendicular to the axis of the lens) instead of the line Ee. For the only error is this: that we assume the convergence of the rays in the perpendicular plane passing through ge to take place at k instead of at e. But the interval ek is proportional to the breadth of the pencil, and we shall always neglect quantities of this order in comparison with quantities, as EM, independent of that breadth. The same reasoning will apply to other cases: and we shall, therefore, without further explanation, assume that a pencil always converges to two lines, one perpen-

dicular to the paper plane, and the other in the paper plane, and perpendicular to the axis of the lens.

Now, to estimate the influence of this kind of convergence on vision with a telescope, let Fig. 5 represent the course of the rays in an astronomical telescope, with the common negative eye-piece, adapted to distinct vision of the center of the field of view, for eyes requiring parallel rays. The rays coming from a point of an object in the axis of the telescope produced, are refracted by the object-glass (which we suppose perfect), received by the first eye-glass, and made to converge accurately to the point F; they then diverge on the second eye-glass, and emerge parallel, and form a distinct image on the retina. But the rays coming from another part of the object pass through some other part of the lens, and are refracted so as to pass through two lines; one hk in the paper plane, which is not generally at the same distance from that lens as the point F; and another perpendicular to the paper plane at g. A similar effect takes place at the next lens: so that the rays of this pencil after emergence are not parallel, but converge to two lines, the convergence of the rays in the paper plane being in most cases more rapid than that of the rays in the perpendicular plane; or, the line perpendicular to the paper plane being generally nearer than that in the paper plane. When the eye, therefore, is turned to receive these rays, they do not converge to a point on the retina, but converge to two lines within the eye, and then diverge upon the retina.

Now, if it were in our power to bring the retina nearer to the crystalline (the effects of which may be exactly imitated by pushing in the eye-piece), so as to form sections of the pencil at different points, supposing the pencil to be circular (as is usually the case), we should have a series of sections similar to those given in Fig. 6. The two lines correspond to the positions of the retina when it is brought up to the lines of convergence; if the pencil be exactly

circular, it will easily be seen that their lengths are equal. All the others are ellipses, with the exception of the one equidistant from the two lines, which is a circle whose diameter is half the length of either line. The image of a point, therefore, on the retina, when viewed out of the center of the field of view, is generally an ellipse, becoming sometimes a circle, and sometimes a straight line.

It may happen that the two lines of convergence intersect each other, in which case, the rays tend, after refraction, to converge to a point. Instead of the images in Fig. 6, we should then have (upon moving the retina) the series in Fig. 7, a series of circles and a point. If this condition then be satisfied, the image of a point viewed out of the center of the field of view, is a circle, but on pushing in the eye-piece it may be made a point, as it ought to be.

We shall now proceed with the mathematical investigations suggested by these considerations.

Prop. III. A pencil of rays, whose axis intersects the axis of the lens, is incident on a lens: to find the distance (from the lens) of the plane perpendicular to the axis of the lens, at which the convergence of rays, in a perpendicular plane, takes place.

Let L, Fig. 4, be the point from which the rays diverge: let LAF be the axis of the incident pencil, and let it be refracted first in the direction FD, and afterwards in the direction GM. Let H and K be the centers of the first and second surfaces; join LH, and produce it to meet the first refracted ray in D: join KD, cutting the emergent ray in M. Now, because LHD is perpendicular to the first surface, all rays in the conical surface, generated by the revolution of LF about LH, will converge to D, and, therefore, rays near each other in a perpendicular plane will, after refraction, converge to D. The refracted rays will, therefore, be in the surface of the cone generated by the revolution of GD about KD;

and, as KD is perpendicular to the second surface, they will, after the second refraction, converge to M. If, then, a plane be drawn through M perpendicular to the axis of the lens, and cutting it in O, OC is the distance which we are to find.

Draw LN, DP, perpendicular to the axis of the lens: as before, let $BF=\alpha$, and let $AB=b$, $CE=c$; also, let $NB=a$, $QD=x$, $CO=z$: and let B, C, A, Z, be the values of b, c, a, z, when $\alpha=0$. Also, let $F=$ focal length of the lens, $f=\frac{1}{F}$,

$$\text{and let } \frac{1}{B}=\frac{f}{2}+e, \text{ then } \frac{1}{C}=\frac{f}{2}-e:$$

$$\text{and let } \frac{1}{A}=\frac{f}{2}+g, \text{ then } \frac{1}{Z}=\frac{f}{2}-g.$$

Let r and s be the radii of the first and second surfaces, and let

$$\frac{1}{r}=\frac{f}{2(n-1)}+v, \text{ then } \frac{1}{s}=\frac{f}{2(n-1)}-v.$$

$$\text{Now, } NL \text{ approximately } = \frac{a-b}{b}\,\alpha;$$

$$\therefore\ BQ=\frac{r}{r+a}\cdot\frac{a-b}{b}\,\alpha;\quad \therefore\ QF=\frac{a(r+b)}{b(r+a)}\,\alpha.$$

Now, by the formula which we have before used,

$$\frac{1}{QD}=\frac{n-1}{nr}-\frac{1}{n\,.\,QL}+\frac{QF^2}{2}\cdot\frac{n-1}{n^3}\cdot\left(\frac{1}{r}+\frac{1}{QL}\right)^2\cdot\left(\frac{1}{r}+\frac{n+1}{QL}\right).$$

$$\text{But } HL=\sqrt{\left\{\overline{a+r}|^2+\overline{\frac{a-b}{b}}\Big|^2\alpha^2\right\}}=a+r+\frac{\overline{a-b}|^2}{b^2(a+r)}\cdot\frac{\alpha^2}{2};$$

$$\therefore\ QL=a+\frac{\overline{a-b}|^2}{b^2(a+r)}\cdot\frac{\alpha^2}{2};\ \therefore\ \frac{1}{QL}=\frac{1}{a}-\frac{\overline{a-b}|^2}{a^2b^2(a+r)}\cdot\frac{\alpha^2}{2}:$$

hence, (neglecting always α^4)

$$\frac{1}{QD}=\frac{n-1}{nr}-\frac{1}{na}+\frac{\overline{a-b}|^2}{na^2b^2(a+r)}\cdot\frac{\alpha^2}{2}$$

$$+\frac{n-1}{n^2}\cdot\left(\frac{1}{r}+\frac{1}{a}\right)^2\cdot\left(\frac{1}{r}+\frac{n+1}{a}\right)\cdot\frac{\left(\frac{1}{b}+\frac{1}{r}\right)^2}{\left(\frac{1}{a}+\frac{1}{r}\right)^2}\cdot\frac{a^2}{2},$$

or putting X for $\dfrac{1}{\frac{n-1}{nr}-\frac{1}{na}}$,

$$\frac{1}{x}=\frac{1}{X}+\frac{a^2}{2}\left\{\frac{\overline{a-b}|^2}{na^2b^2(a+r)}+\frac{n-1}{n^3}\left(\frac{1}{r}+\frac{1}{b}\right)^2\cdot\left(\frac{1}{r}+\frac{n+1}{a}\right)\right\};$$

taking the reciprocal,

$$x=X-\frac{a^2}{2}\cdot X^2\left\{\frac{\overline{a-b}|^2}{na^2b^2(a+r)}+\frac{n-1}{n^3}\left(\frac{1}{r}+\frac{1}{b}\right)^2\cdot\left(\frac{1}{r}+\frac{n+1}{a}\right)\right\}.$$

It is now necessary to find an expression for RD.

$$\text{Now } DP=\frac{x-r}{a+r}\cdot\frac{a-b}{b}a;$$

$$\therefore\ HP=\sqrt{\left\{\overline{x-r}|^2-\frac{\overline{x-r}|^2\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}a^2\right\}}=x-r-\frac{\overline{x-r}\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}\cdot\frac{a^2}{2};$$

$\therefore$ KP (putting t for the thickness of the lens)

$$=x+s-t-\frac{\overline{x-r}\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}\cdot\frac{a^2}{2}.$$

As before, the terms depending on t may be neglected;

$$\therefore\ KP=x+s-\frac{\overline{x-r}\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}\cdot\frac{a^2}{2};$$

$$\therefore\ KD=\sqrt{(KP^2+DP^2)}$$

$$=\sqrt{\left\{\overline{x+s}|^2-\frac{\overline{x+s}\cdot\overline{x-r}\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}a^2+\frac{\overline{x-r}|^2\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}a\right\}}$$

$$=\sqrt{\left\{\overline{x+s}|^2-\frac{\overline{r+s}\cdot\overline{x-r}\cdot\overline{a-b}|^2}{\overline{a+r}|^2\cdot b^2}a^2\right\}}=x+s-\frac{\overline{r+s}\cdot\overline{x-r}\cdot\overline{a-b}|^2}{\overline{x+s}\cdot\overline{a+r}|^2\cdot b^2}\cdot\frac{a^2}{2};$$

$$\therefore\ RD = x - \frac{\overline{r+s}\,.\,\overline{x-r}\,.\,\overline{a-b}|^2}{\overline{x+s}\,.\,\overline{a+r}|^2\,.\,b^2}\,.\,\frac{a^2}{2}$$

$$= X - \frac{a^2}{2}\left\{X^2\left(\frac{\overline{a-b}|^2}{na^2b^2\,(a+r)} + \frac{n-1}{n^3}\left(\frac{1}{r}+\frac{1}{b}\right)^2.\left(\frac{1}{r}+\frac{n+1}{a}\right)\right) + \frac{\overline{r+s}\,.\,\overline{x-r}\,.\,\overline{a-b}|^2}{\overline{x+s}\,.\,\overline{a+r}|^2\,.\,b^2}\right\}$$

Call this $X - Y\frac{a^2}{2}$. Then (in the same manner as in a former Proposition)

$$\frac{1}{RM} = \overline{n-1}\,\frac{1}{s} + \frac{n}{X - Y\frac{a^2}{2}} + n^2\,.\,\overline{n-1}\,.\left(\frac{1}{s}+\frac{1}{x}\right)^2\left(\frac{1}{s}+\frac{n+1}{nx}\right).\frac{GR^2}{2}\,.$$

But $RC = \dfrac{s}{s+x}\,.\,\dfrac{x-r}{a+r}\,.\,\dfrac{a-b}{b}\,a = \dfrac{\left(\frac{1}{r}-\frac{1}{x}\right)\left(\frac{1}{b}-\frac{1}{a}\right)}{\left(\frac{1}{x}+\frac{1}{s}\right)\left(\frac{1}{r}+\frac{1}{a}\right)}\,a,$

$$\text{or} = \frac{\left(\frac{1}{r}-\frac{1}{x}\right)\left(\frac{1}{B}-\frac{1}{A}\right)}{\left(\frac{1}{x}+\frac{1}{s}\right)\left(\frac{1}{r}+\frac{1}{A}\right)}\,a\,;$$

which, observing that $\frac{1}{x} = \frac{n-1}{nr} - \frac{1}{nA}$ nearly,

or $= \frac{1}{nZ} - \frac{n-1}{ns}\left(\text{since } \frac{1}{A}+\frac{1}{Z} = f = \overline{n-1}\left(\frac{1}{r}+\frac{1}{s}\right)\right)$, gives

$$RC = \frac{\frac{1}{B}-\frac{1}{A}}{\frac{1}{s}+\frac{1}{Z}}\,a\,;$$

$$\therefore\ GR = a - \frac{\frac{1}{B}-\frac{1}{A}}{\frac{1}{s}+\frac{1}{Z}}\,a = a\,\frac{\frac{1}{s}+\frac{1}{Z}+\frac{1}{A}-\frac{1}{B}}{\frac{1}{s}+\frac{1}{Z}} = a\,\frac{\frac{1}{s}+\frac{1}{C}}{\frac{1}{s}+\frac{1}{Z}}\,.$$

Hence $\frac{1}{RM} = \overline{n-1}\,\frac{1}{s} + \frac{n}{X} + \frac{nY}{X^2}\cdot\frac{a^2}{2}$

$$+ n^2.\overline{n-1}\left(\frac{1}{s}+\frac{1}{x}\right)^2\left(\frac{1}{s}+\frac{n+1}{nx}\right)\cdot\frac{\left(\frac{1}{s}+\frac{1}{C}\right)^2}{\left(\frac{1}{s}+\frac{1}{\mathcal{Z}}\right)^2}\cdot\frac{a^2}{2}$$

$$= \overline{n-1}\,\frac{1}{s}+\frac{n}{X}+\frac{nY}{X^2}\cdot\frac{a^2}{2}+\frac{n-1}{n^2}\cdot\left(\frac{1}{s}+\frac{1}{C}\right)^2\cdot\left(\frac{1}{s}+\frac{n+1}{\mathcal{Z}}\right)\cdot\frac{a^2}{2};$$

which, since

$$\overline{n-1}\,\frac{1}{s}+\frac{n}{X}=\frac{\overline{n-1}}{s}+\frac{\overline{n-1}}{r}-\frac{1}{a}=\frac{1}{F}-\frac{1}{a}=\frac{1}{F}-\frac{1}{A}+\frac{a-A}{A^2}=\frac{1}{\mathcal{Z}}+\frac{a-A}{A^2},$$

gives $\frac{1}{RM}=\frac{1}{Z}+\frac{a-A}{A^2}+\frac{nY}{X^2}\cdot\frac{a^2}{2}+\frac{n-1}{n^2}\left(\frac{1}{s}+\frac{1}{C}\right)^2\left(\frac{1}{s}+\frac{n+1}{\mathcal{Z}}\right)\cdot\frac{a^2}{2}.$

$$\text{whence } RM=\mathcal{Z}-\frac{\mathcal{Z}^2}{A^2}(a-A)-\frac{n\mathcal{Z}^2}{X^2}\cdot Y\frac{a^2}{2}$$

$$-\frac{n-1}{n^2}\mathcal{Z}^2\left(\frac{1}{s}+\frac{1}{C}\right)^2\cdot\left(\frac{1}{s}+\frac{n+1}{\mathcal{Z}}\right)\cdot\frac{a^2}{2}:$$

which, putting for Y its value, and substituting for x in the term

$$\frac{\overline{r+s}\,.\,\overline{x-r}\,.\,\overline{a-b}|^2}{\overline{x+s}\,.\,\overline{a+r}|^2\,.\,b^2x^2},$$

$$\text{gives } RM=\mathcal{Z}-\frac{\mathcal{Z}^2}{A^2}(a-A)$$

$$-\mathcal{Z}^2\left\{\frac{\left(\frac{1}{b}-\frac{1}{a}\right)^2}{a+r}+\frac{n-1}{n^2}\left(\overline{\frac{1}{r}+\frac{1}{B}}\Big|^2\cdot\overline{\frac{1}{r}+\frac{n-1}{A}}+\overline{\frac{1}{s}+\frac{1}{C}}\Big|^2\cdot\overline{\frac{1}{s}+\frac{n+1}{\mathcal{Z}}}\right)\right.$$

$$\left.+n\,\frac{\left(\frac{1}{s}+\frac{1}{r}\right)\left(\frac{1}{B}-\frac{1}{A}\right)^2}{\left(\frac{1}{s}+\frac{1}{Z}\right)\left(\frac{1}{r}+\frac{1}{A}\right)}\cdot\left(\frac{n-1}{nr}-\frac{1}{nA}\right)\left(\frac{1}{n\mathcal{Z}}-\frac{n-1}{ns}\right)\right\}\frac{a^2}{2}.$$

Now $OK=\sqrt{(KM^2-OM^2)}=KM-\dfrac{OM^2}{2\,KM}=s+RM-\dfrac{OM^2}{2\,(s+z)}$;

$$\therefore\ CO=RM-\frac{OM^2}{2\,(s+z)}.\quad \text{And } RC=\frac{\frac{1}{B}-\frac{1}{A}}{\frac{1}{s}+\frac{1}{z}}\,a;$$

$$\therefore\ OM=\frac{s+z}{s}\,RC=z\left(\frac{1}{B}-\frac{1}{A}\right)a;$$

$$\therefore\ CO=RM-z^2\,\frac{\left(\frac{1}{B}-\frac{1}{A}\right)^2}{s+z}\cdot\frac{a^2}{2}=RM-z^2.\,\frac{\left(\frac{1}{C}-\frac{1}{z}\right)^2}{z+s}\cdot\frac{a^2}{2}:$$

and, putting for RM its value,

$$CO=z-\frac{z^2}{A^2}\,(a-A)$$

$$-\,z^2\left\{\frac{\left(\frac{1}{B}-\frac{1}{A}\right)^2}{A+r}+\frac{\left(\frac{1}{C}-\frac{1}{z}\right)^2}{z+s}+\frac{1}{n}\cdot\frac{\left(\frac{1}{s}+\frac{1}{r}\right)\left(\frac{1}{B}-\frac{1}{A}\right)\left(\frac{1}{z}-\frac{1}{C}\right)}{\left(\frac{1}{s}+\frac{1}{z}\right)\left(\frac{1}{r}+\frac{1}{A}\right)}\right.$$

$$\times\left(\frac{n-1}{r}-\frac{1}{A}\right)\left(\frac{1}{z}-\frac{n-1}{s}\right)+\frac{n-1}{n^2}\left(\overline{\frac{1}{r}+\frac{1}{B}}\right|^2\cdot\overline{\frac{1}{r}+\frac{n+1}{A}}$$

$$\left.+\,\overline{\frac{1}{s}+\frac{1}{C}}\,\right|^2\cdot\overline{\frac{1}{s}+\frac{n+1}{z}}\Bigg)\right\}\frac{a^2}{2}.$$

The expression within the brackets, it will be observed, is strictly symmetrical with respect to the quantities on each side of the lens.

If, in the expression for CO or z we substitute for

$$\frac{1}{A},\ \frac{1}{B},\ \frac{1}{C},\ \frac{1}{z},\ \frac{1}{r},\ \frac{1}{s},$$

the values above given, we find

$$x = Z - \frac{Z^2}{A^2}(a - A)$$

$$- Z^2 f \left\{ \frac{\overline{e-g}|^2}{n} + \frac{n^2 f^2}{4(n-1)^2} + \overline{e+v}|^2 + \frac{2}{n} . \overline{e+v}\,(\overline{n+1}\,.g + v)\right\} . \frac{a^2}{2}.$$

If NL, the breadth of the first image, be called β, OM, the breadth of the second, β', then

$$\beta = \frac{a-b}{b}\,\alpha = A\,(e-g)\,\alpha, \quad \text{or } \alpha = \frac{\beta}{A\,(e-g)}:$$

$$\text{and } \beta' = Z\,(e-g)\,.\,\alpha, \quad \text{or } \alpha = \frac{\beta'}{Z\,(e-g)}.$$

Substituting these values for α,

$$z = Z - \frac{Z^2}{A^2}(a - A) - \frac{Z^2}{A^2} . \left\{ \frac{f}{n} \right.$$

$$\left. + \frac{f}{(e-g)^2}\left(\frac{n+2}{n}v^2 + \frac{2n+2}{n} . \overline{e+g} . v + \frac{2n+2}{n} eg + e^2 + \frac{n^2 f^2}{4(n-1)^2}\right)\right\} \frac{\beta^2}{2}.$$

$$\text{Or } z = Z - \frac{Z^2}{A^2}(a - A) - \left\{ \frac{f}{n} \right.$$

$$\left. + \frac{f}{(e-g)^2}\left(\frac{n+2}{n}v^2 + \frac{2n+2}{n} . \overline{e+g} . v + \frac{2n+2}{n} eg + e^2 + \frac{n^2 f^2}{4(n-1)^2}\right)\right\} \frac{\beta'^2}{2}.$$

We shall denote the quantity

$$\frac{f}{(e-g)^2}\left(\frac{n+2}{n}v^2 + \frac{2n+2}{n} . \overline{e+g} . v + \frac{2n+2}{n} eg + e^2 + \frac{n^2 f^2}{4(n-1)^2}\right),$$

by the letter V, and shall put U for $\frac{f}{n} + V$: hence, we have

$$z = Z - \frac{Z^2}{A^2}(a - A) - Z^2 . \overline{e-g}|^2 . U . \frac{\alpha^2}{2},$$

$$z = Z - \frac{Z^2}{A^2}(a - A) - \frac{Z^2}{A^2} U . \frac{\beta^2}{2},$$

$$z = \mathfrak{z} - \frac{\mathfrak{z}^2}{A^2}(a - A) - U.\frac{\beta''^2}{2},$$

$$\text{where} \quad U = \frac{f}{n} + V.$$

It will be observed that $\mathfrak{z} - \frac{\mathfrak{z}^2}{A^2}(a - A)$ enters into these expressions as equivalent to $\dfrac{1}{\frac{1}{F} - \frac{1}{a}}$.

The expression fails when $A = F$, which makes $\mathfrak{z}$ infinite. We might for this case give a different form to the latter part of the investigation, but the following method will be more convenient. Suppose the direction of the rays reversed: then (as the quantity U is the same in whatever direction the rays be supposed incident, and as β is then the breadth of the second image)

$$a = \frac{1}{\frac{1}{F} - \frac{1}{z}} - U.\frac{\beta^2}{2}:$$

$$\text{hence } \frac{1}{a} = \frac{1}{F} - \frac{1}{z} + \left(\frac{1}{F} + \frac{1}{z}\right)^2.U.\frac{\beta^2}{2} = \frac{1}{F} - \frac{1}{z} + \frac{1}{F^2}.U\frac{\beta^2}{2},$$

$\left(\text{since } \frac{1}{z} \text{ would be o if there was no aberration}\right)$: therefore

$$\frac{1}{z} = \frac{1}{F} - \frac{1}{a} + \frac{1}{F^2}.U\frac{\beta^2}{2} = \frac{1}{F} - \frac{1}{F + a - A}$$

$$+ \frac{1}{F^2}.U.\frac{\beta^2}{2} = \frac{a - A}{F^2} + \frac{1}{F^2}.U.\frac{\beta^2}{2}.$$

The rays converge, therefore, after refraction to a line whose distance is

$$\frac{F^2}{a - A + U\frac{\beta^2}{2}}.$$

PROP. IV. To determine the distance (from the lens) of the plane perpendicular to its axis at which the convergence of rays in the paper plane takes place.

Let Lf, fg, gm, be the course of a ray near LF, and let it intersect GM at S, and draw ST perpendicular to the axis of the lens: then, CT is the distance required. Let $CT=\zeta$, $Ff=\delta a$. Then, to determine Qd, we must use the same value of LQ as before, but instead of QF we must use $QF+\delta a$; whence

$$\frac{1}{Qd}=\frac{n-1}{nr}-\frac{1}{na}+\frac{\overline{a-b}|^2}{na^2b^2(a+r)}\cdot\frac{a^2}{2}$$

$$+\frac{n-1}{n^3}\left(\frac{1}{r}+\frac{1}{a}\right)^2\cdot\left(\frac{1}{r}+\frac{n+1}{a}\right)\cdot\left(\frac{\frac{1}{b}+\frac{1}{r}}{\frac{1}{a}+\frac{1}{r}}a+\delta a\right)^2\cdot\frac{1}{2};$$

or, neglecting $\overline{\delta a}|^2$,

$$\frac{1}{Qd}=\frac{1}{x}+\frac{n-1}{n^3}\left(\frac{1}{r}+\frac{1}{a}\right)\cdot\left(\frac{1}{r}+\frac{1}{b}\right)\cdot\left(\frac{1}{r}+\frac{n+1}{a}\right).a\delta a;$$

$$\therefore\ Qd=x-\frac{n-1}{n^3}.x^2\left(\frac{1}{r}+\frac{1}{a}\right)\cdot\left(\frac{1}{r}+\frac{1}{b}\right)\cdot\left(\frac{1}{r}+\frac{n+1}{a}\right).a\delta a;$$

$$\therefore\ Dd=\frac{n-1}{n^3}x^2\left(\frac{1}{r}+\frac{1}{a}\right)\left(\frac{1}{r}+\frac{1}{b}\right)\left(\frac{1}{r}+\frac{n+1}{a}\right).a\delta a.$$

Now, $Dd : De :: \sin geR : \sin gdQ :: GR : FQ$ nearly

$$::\ a\frac{\frac{1}{s}+\frac{1}{C}}{\frac{1}{s}+\frac{1}{Z}} : a\frac{\frac{1}{r}+\frac{1}{b}}{\frac{1}{r}+\frac{1}{a}};$$

$$\therefore\ De=Dd\times\frac{\left(\frac{1}{r}+\frac{1}{b}\right)\left(\frac{1}{s}+\frac{1}{Z}\right)}{\left(\frac{1}{r}+\frac{1}{a}\right)\left(\frac{1}{s}+\frac{1}{C}\right)}=\frac{n-1}{n^3}x^2.\left(\frac{1}{r}+\frac{1}{b}\right)^2.\frac{\frac{1}{s}+\frac{1}{Z}}{\frac{1}{s}+\frac{1}{C}}.\left(\frac{1}{r}+\frac{n+1}{a}\right)a\delta a.$$

For refraction at the second surface, therefore, instead of using

$$DR = X - Y\frac{a^2}{2},$$

we must use

$$eR = X - Y\frac{a^2}{2} - \frac{n-1}{n^3}x^2\left(\frac{1}{r}+\frac{1}{b}\right)^2 \cdot \frac{\frac{1}{s}+\frac{1}{\mathfrak{Z}}}{\frac{1}{s}+\frac{1}{C}} \cdot \left(\frac{1}{r}+\frac{n+1}{a}\right) a\delta a;$$

and instead of using

$$RG = a\,\frac{\frac{1}{s}+\frac{1}{C}}{\frac{1}{s}+\frac{1}{\mathfrak{Z}}},$$

we must use

$$Rg = a\,\frac{\frac{1}{s}+\frac{1}{C}}{\frac{1}{s}+\frac{1}{\mathfrak{Z}}} + \delta a.$$

From this $\frac{1}{Rm} = \overline{n-1}\,\frac{1}{s} + \frac{n}{X} + \frac{nY}{X^2}\cdot\frac{a^2}{2}$

$$+ \frac{n-1}{n^2}\cdot\left(\frac{1}{r}+\frac{1}{b}\right)^2 \cdot \frac{\frac{1}{s}+\frac{1}{\mathfrak{Z}}}{\frac{1}{s}+\frac{1}{C}} \cdot \left(\frac{1}{r}+\frac{n+1}{a}\right) a\delta a$$

$$+ n^2.\overline{n-1}\left(\frac{1}{s}+\frac{1}{x}\right)^2 \cdot \left(\frac{1}{s}+\frac{n+1}{nx}\right) \cdot \left(\frac{\frac{1}{s}+\frac{1}{C}}{\frac{1}{s}+\frac{1}{\mathfrak{Z}}}\,a + \delta a\right)^2 \cdot \frac{1}{2}$$

$$= \frac{1}{RM} + \frac{n-1}{n^2}\left(\frac{1}{r}+\frac{1}{b}\right)^2 \cdot \frac{\frac{1}{s}+\frac{1}{\mathfrak{Z}}}{\frac{1}{s}+\frac{1}{C}} \cdot \left(\frac{1}{r}+\frac{n+1}{a}\right) a\delta a$$

$$+ n^2 . \overline{n-1} . \left(\frac{1}{s} + \frac{1}{x}\right)^2 . \left(\frac{1}{s} + \frac{n+1}{nx}\right) . \frac{\frac{1}{s} + \frac{1}{C}}{\frac{1}{s} + \frac{1}{Z}} a \delta a :$$

taking the reciprocal and subtracting it from RM, and substituting as before for x,

$$Mm = Z^2 . \frac{n-1}{n^2} \left\{ \left(\frac{1}{r} + \frac{1}{b}\right)^2 . \frac{\frac{1}{s} + \frac{1}{Z}}{\frac{1}{s} + \frac{1}{C}} \left(\frac{1}{r} + \frac{n+1}{a}\right) \right.$$

$$\left. + \left(\frac{1}{s} + \frac{1}{Z}\right) \left(\frac{1}{s} + \frac{1}{C}\right) \left(\frac{1}{s} + \frac{n+1}{Z}\right) \right\} a \delta a.$$

Now $Mm : MS :: \sin MSm : \sin SmR :: Gg : RG$ nearly

$$:: \delta a : a \frac{\frac{1}{s} + \frac{1}{C}}{\frac{1}{s} + \frac{1}{Z}} ;$$

$$\therefore MS = Mm . \frac{\frac{1}{s} + \frac{1}{C}}{\frac{1}{s} + \frac{1}{Z}} . \frac{a}{\delta a}$$

$$= Z^2 . \frac{n-1}{n^2} . \left\{ \left(\frac{1}{r} + \frac{1}{b}\right)^2 . \left(\frac{1}{r} + \frac{n+1}{a}\right) + \left(\frac{1}{s} + \frac{1}{C}\right)^2 . \left(\frac{1}{s} + \frac{n+1}{Z}\right) \right\} a^2$$

$$= Z^2 . \overline{e-g}|^2 . V . a^2.$$

And OT may be considered equal to MS: putting, therefore, for OC its value,

$$\zeta = z - Z^2 . \overline{e-g}|^2 . V . a^2 = Z - \frac{Z^2}{A^2} (a-A) - Z^2 . \overline{e-g}|^2 \left\{ \frac{f}{n} + 3V \right\} \frac{a^2}{2} .$$

We shall put Υ for $\frac{f}{n} + 3V$: hence, we have

$$\zeta = \mathcal{Z} - \frac{\mathcal{Z}^2}{A^2}(a-A) - \mathcal{Z}^2 . \overline{e-g}^2 . \Upsilon \frac{a^2}{2},$$

$$\zeta = \mathcal{Z} - \frac{\mathcal{Z}^2}{A^2}(a-A) - \frac{\mathcal{Z}^2}{A^2} . \Upsilon . \frac{\beta^2}{2},$$

$$\zeta = \mathcal{Z} - \frac{\mathcal{Z}^2}{A^2}(a-A) - \Upsilon . \frac{\beta'^2}{2},$$

where $\Upsilon = \frac{f}{n} + 3V$.

If the rays in the paper-plane diverge from the distance η instead of a, the value of η when $a = 0$ being A, we have merely to put in the forms above, η for a.

This expression, like the former, fails when $A = F$. By the same kind of investigation as in the last Proposition, it is found that the rays after refraction converge to a line at the distance $\dfrac{F^2}{\eta - A + \Upsilon \frac{\beta^2}{2}}$.

It appears, then, (supposing $a - A = 0$, and $\eta - A = 0$),

1st. That the rays will converge accurately to a point if $z - \zeta = 0$, or if $V = 0$.

2d. That the whole image will be in the same plane if at the same time $\mathcal{Z} - z = 0$, or if $\frac{f}{n} + V = 0$.

3d. That these conditions are incompatible: and, therefore, when rays proceed from an object, or a distinct image in one plane, they cannot, after refraction by a lens, form a distinct image in another plane.

4th. If the breadth of the pencil, supposed circular, be λ, the rays will meet the plane perpendicular to the axis of the lens at the distance $\mathfrak{Z}$, in an ellipse, of which the axes are $\lambda \frac{U}{\mathfrak{Z}} \cdot \frac{\beta'^2}{2}$, and $\lambda \frac{\Upsilon}{\mathfrak{Z}} \cdot \frac{\beta'^2}{2}$: and when $V=0$ or $U=\Upsilon$, the ellipse becomes a circle: its diameter is

$$\frac{\lambda}{\mathfrak{Z}} \cdot \frac{f}{n} \cdot \frac{\beta'^2}{2}.$$

5th. The rays will meet the plane at the distance $\mathfrak{Z}-p$ in an ellipse whose axes are

$$\frac{\lambda}{\mathfrak{Z}} \left(U \frac{\beta'^2}{2} - p\right), \quad \text{and } \frac{\lambda}{\mathfrak{Z}} \left(\Upsilon \frac{\beta'^2}{2} - p\right).$$

If $p = U . \frac{\beta'^2}{2}$, or if $p = \Upsilon \frac{\beta'^2}{2}$, the ellipse becomes a straight line.

$$\text{If } p = \tfrac{1}{2}(U+\Upsilon) \frac{\beta'^2}{2} = \left(\frac{f}{n} + 2V\right) \frac{\beta'^2}{2},$$

the ellipse becomes a circle whose diameter $= \frac{\lambda}{\mathfrak{Z}} V . \frac{\beta'^2}{2}$.

If $V=0$, the ellipse becomes a circle, whatever be the value of p; the diameter is

$$\frac{\lambda}{\mathfrak{Z}} \left(\frac{f}{n} \cdot \frac{\beta'^2}{2} - p\right).$$

If at the same time $p = \frac{f}{n} \cdot \frac{\beta'^2}{2}$, the diameter is 0, or the circle becomes a point, and a distinct image is formed.

6th. If $V + \frac{f}{2n} = 0$, the image formed on the plane at the distance $\mathfrak{Z}$ will be circular. Its diameter will be

$$\frac{\lambda}{\mathfrak{Z}} V . \frac{\beta'^2}{2}, \quad \text{or } \frac{\lambda}{\mathfrak{Z}} \cdot \frac{f}{2n} \cdot \frac{\beta'^2}{2}.$$

Ex. 1. A pencil of rays diverging from a point of an object, in a plane perpendicular to the axis of a lens, passes through the center of the lens: to find the distances of the planes perpendicular to the axis passing through the lines of convergence.

Here $\frac{1}{B}$ is infinite, or e is infinite, and, therefore, in V we must neglect all other quantities in comparison with e. This reduces its expression to f:

$$\text{therefore } U = \frac{n+1}{n} f, \quad \Upsilon = \frac{3n+1}{n} f.$$

$$\text{Therefore } z = \mathfrak{Z} - \frac{n+1}{n} f . \frac{\beta^2}{2}, \quad \zeta = \mathfrak{Z} - \frac{3n+1}{n} f . \frac{\beta'^2}{2}.$$

The convergence of rays in the paper-plane, therefore, takes place in a spherical surface, whose radius is $\frac{nF}{3n+1}$, and that of rays in a perpendicular plane in a spherical surface, whose radius is $\frac{nF}{n+1}$. It is remarkable that these radii depend only on the focal length, and not on the curvatures of the surfaces of the lens, or the situation of the object.

Ex. 2. For the camera obscura, suppose a single lens to be used, with a diaphragm at a distance from the lens to limit the pencil of rays: to find the situation of the diaphragm, and the radii of the surfaces most favourable to the formation of a distinct image.

Here e and v are the unknown quantities: since $\frac{1}{A} = 0$, $\therefore g = -\frac{f}{2}$: and the expression for V becomes

$$\frac{f}{\left(e+\frac{f}{2}\right)^2}\left(\frac{n+2}{n} v^2 + \frac{2n+2}{n} . \overline{e - \frac{f}{2}} . v - \frac{n+1}{n} fe + e^2 + \frac{n^2 f^2}{4(n-1)^2}\right).$$

Now, since it is impossible to form an image which will be perfectly distinct, we must consider what kind of aberration is least disagreeable to the eye. I believe that the confusion occasioned by making the image of every point a circle is less disagreeable than that occasioned by making the image an ellipse, or straight line. Now, this can be effected in two ways: either by making $V+\frac{f}{2n}=0$, or by making $V=0$. The former equation is not possible. The latter gives

$$e=-\frac{n+1}{n}\left(v-\frac{f}{2}\right)\pm\frac{1}{n}\sqrt{\left\{\overline{v-\frac{2n^2-1}{2(n-1)}f}\cdot\overline{v+\frac{f}{2(n+1)}}\right\}},$$

and the two values of e are possible, if v be

$$>\frac{2n^2-1}{2(n-1)}\cdot f,\quad \text{or} <-\frac{f}{2(n-1)}.$$

As an example, suppose $v=\frac{2n^2-1}{2(n-1)}f$. The radical disappears, and there is, then, but one value of e, namely,

$$-\frac{n+1}{n}\left(\frac{2n^2-1}{2(n-1)}f-\frac{f}{2}\right)=-\frac{f}{2}\cdot\frac{\overline{n+1}\cdot\overline{2n-1}}{n-1}.$$

Finding from this the value of B, it is discovered to be negative, and, therefore, a diaphragm cannot be placed to receive the rays before they are incident on the lens. It may, however, be placed to receive them after their emergence. The value of C is found to be

$$\frac{n-1}{n^2+n-1}F.\quad \text{Also } r=\frac{n-1}{n^2}F,\quad s=-\frac{1}{n+1}F.$$

The second surface is, therefore, concave. This construction is represented in Fig. 8. If we had taken $v>\frac{2n^2-1}{2(n-1)}f$, we should have found two positions for the diaphragm, both below the lens.

Again, suppose

$$v = -\frac{f}{2(n-1)}. \quad \text{Here } e = \frac{n+1}{2(n-1)}f:$$

$$\text{whence } B = \frac{n-1}{n}F, \quad r = \infty, \quad s = \overline{n-1}.F,$$

or the lens is plano-convex. This construction is represented in Fig. 9: and for this case an easy geometrical demonstration may be given. If v be $< -\frac{f}{2(n-1)}$, there are two positions of the diaphragm equally good, both above the lens.

These constructions, it will be observed, are found by making $V=0$. Now, we have remarked that when $V=0$, the rays of the pencil converge to a point nearer to the lens than the plane at the distance Z, by the space $\frac{f}{n}.\frac{\beta^2}{2}$. If, then, the bottom of the camera obscura, instead of being a plane, were a concave spherical surface, whose radius $= nF$ (which would make the elevation of every point above the plane nearly $= \frac{f}{n}.\frac{\beta^2}{2}$), the image would be accurately formed upon it. It is remarkable that in all the varieties of form of the lens, and position of the diaphragm, which make the above equations possible, the radius of the spherical surface proper to receive the image, continues the same.

This is the theory of Dr. Wollaston's periscopic camera obscura; and, supposing the direction of the rays reversed, it applies also to his periscopic spectacles.

Prop. V. To find the effect of the aberration of the lenses of an eye-piece on the distinctness of an object seen through a telescope.

We may consider the image formed by the object-glass as being perfectly plane and equally distinct in every part. Considering, then, first, the rays in a perpendicular plane, we have $a - A = 0$;

$$\therefore\ z = Z - U . \frac{\beta^2}{2}.$$

Now $z + a'$ = interval between 1st and 2d lenses $= Z + A'$;

$$\therefore\ a' - A' = Z - z = U . \frac{\beta^2}{2}.$$

$$\text{Hence, } z' = Z' - \frac{Z'^2}{A'^2}(a' - A') - U' . \frac{\beta'^2}{2} = Z' - \frac{Z'^2}{A'^2} U . \frac{\beta^2}{2} - U' \frac{\beta'^2}{2}.$$

But it is well known that

$$\frac{\beta''}{\beta'} = \frac{Z'}{A'};\quad \therefore\ z' = Z' - (U + U')\frac{\beta''^2}{2}.$$

$$\text{Hence, } a'' - A'' = Z' - z' = (U + U')\frac{\beta''^2}{2},$$

and hence, in the same manner,

$$z'' = Z'' - (U + U' + U'')\frac{\beta'''^2}{2}.$$

Suppose, now, the eye-piece to consist of four lenses (our reasoning will apply equally to any number). That the center of the last image may be distinctly visible, it must be in the focus of the fourth eye-glass, or A''' must $= F'''$. In this case we have for the distance to which the rays converge after refraction,

$$\frac{F'''^2}{a''' - A''' + U''' \frac{\beta'''^2}{2}}.$$

$$\text{Here } a''' - A''' = Z'' - z'' = (U + U' + U'')\frac{\beta'''^2}{2}:$$

and hence the rays converge after refraction to a line at the distance

$$\frac{F'''^2}{(U+U'+U''+U''')\,\frac{\beta'''^2}{2}}.$$

Suppose these to be received by the eye, which is adapted to parallel rays. The eye may be correctly represented by a convex lens of invariable focal length K: the rays then would converge to a line at the distance from this lens

$$\frac{1}{\frac{1}{K}+\frac{U+U'+U''+U'''}{F'''^2}\cdot\frac{\beta'''^2}{2}} = K - \frac{K^2}{F'''^2}(U+U'+U''+U''')\,\frac{\beta'''^2}{2}.$$

But the distance of the retina is K. Hence, the rays intersect at the distance

$$\frac{K^2}{F'''^2}(U+U'+U''+U''')\,\frac{\beta'''^2}{2},$$

before meeting the retina: and if λ be the breadth of the pencil which enters the eye, the extent of the diffusion on the retina is

$$\lambda\cdot\frac{K}{F'''^2}(U+U'+U''+U'''^2)\,\frac{\beta'''^2}{2},$$

and the angle which this subtends at the center of the crystalline is

$$\frac{\lambda}{F'''^2}(U+U'+U''+U''')\,\frac{\beta'''^2}{2}.$$

Now, let L be the aperture of the object-glass; p the magnifying power of the telescope (an abstract number); θ the angular distance of the object from the center of the field of view, not magnified. Then (by the known properties of telescopes), $\frac{\beta'''}{F'''}$ = tan. apparent distance of the object from the center of the field (as magnified) $= p\theta$; and $\lambda = \frac{L}{p}$. Substituting these, we finally obtain for the angular extent of the diffusion,

$$\frac{pL\theta^2}{2}(U+U'+U''+U''').$$

In this form the diffusion can be numerically calculated.

In the same manner, putting η for a, and Υ for U, we find, for the apparent extent of the diffusion in the paper-plane,

$$\frac{pL\theta^2}{2}(\Upsilon+\Upsilon'+\Upsilon''+\Upsilon''').$$

Hence, the appearance presented by a point is an ellipse, the apparent angular extents of whose axes are respectively

$$\frac{pL\theta^2}{2}(U+U'+U''+U'''), \quad \text{and} \quad \frac{pL\theta^2}{2}(\Upsilon+\Upsilon'+\Upsilon''+\Upsilon'''),$$

and of which the real angular extents are, therefore,

$$\frac{L\theta^2}{2}\Sigma(U), \quad \text{and} \quad \frac{L\theta^2}{2}\Sigma(\Upsilon).$$

These we shall call k and κ.

1st. Since $U = \frac{f}{n} + V$, and $\Upsilon = \frac{f}{n} + 3V$, we have

$$3U - \Upsilon = \frac{2f}{n},$$

a constant depending only on the focal length of the lens. A similar equation is true for every lens: adding all

$$3\Sigma(U) - \Sigma(\Upsilon) = \Sigma\left(\frac{2f}{n}\right).$$

Multiplying by $\frac{L\theta^2}{2}$,

$$3k - \kappa = \frac{L\theta^2}{n}\Sigma(f).$$

2d. Since, in the common eye-pieces, f, f', &c. are all positive, it is impossible to make k and κ vanish at the same time. This appears to be an insuperable obstacle to the improvement of eye-pieces.

3d. By assuming for $\Sigma(V)$ different values, we may form a series of ellipses representing the different appearances of a point in a given part of the field of view, and may select that which appears least offensive. In this manner, Fig. 10 was constructed*; the values of $\Sigma(V)$ are placed below. It will be observed that this series of figures is essentially different from that in Figs. 6 and 7: the latter are sections of the same pencil at different distances from the lens, while those under consideration are sections of different pencils (produced by altering the curvatures of the lenses without altering their focal lengths) at the same distance from the lens which we take as representing the eye. Thus, in Fig. 6, the two straight lines are equal in length; in Fig. 10, the line corresponding to

$$\Sigma(V) = -\Sigma\left(\frac{f}{n}\right),$$

is three times as long as that corresponding to

$$\Sigma(V) = -\tfrac{1}{3}\Sigma\left(\frac{f}{n}\right).$$

4th. It appears to me that the confusion least disagreeable is that corresponding to

$$\Sigma(V) = -\tfrac{1}{2}\Sigma\left(\frac{f}{n}\right);$$

or, if another is to be preferred, the negative multiple of $\Sigma\left(\frac{f}{n}\right)$ is

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* It must be observed that Fig. 10 represents the appearances of a point seen at the top or bottom of the field of view. If each of the ellipses be turned through an angle of 90°, it will represent the appearance of a point seen at the right or left side of the field.

to be smaller rather than larger. The best construction, then, will be found by making

$$\Sigma(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0, \quad \text{or } \Sigma(V) + \frac{11}{24}\Sigma\left(\frac{f}{n}\right) = 0.$$

5th. But there is one advantage in the eye-piece satisfying the condition $\Sigma(V) = 0$. Since the rays converge accurately to a point, by bringing the retina nearer to the crystalline, or (which produces the same effect) by pushing in the eye-piece, we shall have the series of images in Fig. 9, and shall, therefore, have a point. The space occupied by the diffusion is always circular. And the equation $\Sigma(V) = 0$ can be satisfied in many cases where the equation

$$\Sigma(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0$$

is impossible. Whenever, then, the last equation cannot be satisfied, or nearly satisfied, it will be best to make $\Sigma(V) = 0$.

6th. If neither of these can be satisfied, $\Sigma(V)$ is essentially positive, and we must, for the most advantageous construction, make $\Sigma(V)$ a minimum.

Ex. 1. Let the eye-glass be a single lens.

Here $C = F$, $e = -\frac{f}{2}$; $A = F$, $g = \frac{f}{2}$, and

$$V = \frac{1}{f}\left(\frac{n+2}{n}v^2 + \frac{3n-2}{4n(n-1)^2}f^2\right).$$

1st. Neither of the above equations is possible, and we must, therefore, make V a minimum. This gives $v = 0$, or the lens is equi-convex.

Then, $V = \frac{3n-2}{4n(n-1)^2}f,$

which, when $n = 1{,}5$, is $\frac{5}{3}f$. And

$$U = \frac{f}{n} + V = \frac{7}{3}f = \frac{7}{3F};$$

$$\Upsilon = \frac{f}{n} + 3V = \frac{17}{3F}.$$

2d. If the lens be plano-convex, in either position, $v = \pm f$, and $V = 4f$; whence

$$U = \frac{14}{3F}, \quad \Upsilon = \frac{38}{3F}.$$

Ex. 2. Suppose two lenses of equal focal length (M) are placed in contact.

Here $C = M$, $B' = -M$, $C' = \frac{M}{2}$; $e = -\frac{m}{2}$, $e' = -\frac{3m}{2}$;

$$A = \frac{M}{2}, \; A' = M; \quad g = \frac{3m}{2}, \; g' = \frac{m}{2}.$$

If $n = 1{,}5$, the expression for V is

$$\frac{f}{(e-g)^2}\left\{\frac{7}{3}v^2 + \frac{10}{3}.\overline{e+g}.v + \frac{10}{3}eg + e^2 + \frac{9}{4}f^2\right\}.$$

Here then $V = \frac{7}{12m}\left(v + \frac{5}{7}m\right)^2 - \frac{25}{84}m$:

$$V' = \frac{7}{12m}\left(v' - \frac{5}{7}m\right)^2 + \frac{17}{84}m:$$

$$\Sigma(V) = \frac{7}{12m}\left(\overline{v + \frac{5}{7}m}\,^2 + \overline{v' - \frac{5}{7}m}\,^2\right) - \frac{2}{21}m.$$

And $\Sigma\left(\frac{f}{n}\right) = \frac{4}{3}m.$

1st. The equation

$$\Sigma\,(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0$$

is impossible.

The values which approach nearest to satisfying this condition are

$$v = -\frac{5}{7}\,m, \quad v' = \frac{5}{7}\,m;$$

which give

$$r = \frac{7}{2}\,M, \quad s = \frac{7}{12}\,M, \quad r' = \frac{7}{12}\,M, \quad s' = \frac{7}{2}\,M:$$

then $\Sigma\,(V) = -\dfrac{2}{21}\,m, \quad \Sigma\,(U) = \dfrac{26}{21\,M}, \quad \Sigma\,(\Upsilon) = \dfrac{22}{21\,M}.$

2d. The equation $\Sigma\,(V) = 0$, or

$$\overline{v + \frac{5}{7}\,m}\,\Big|^2 + \overline{v' - \frac{5}{7}\,m}\,\Big|^2 - \frac{8}{49}\,m^2 = 0,$$

may be satisfied in an infinite number of ways. Thus, suppose

$$\overline{v + \frac{5}{7}\,m}\,\Big|^2 = \overline{v' - \frac{5}{7}\,m}\,\Big|^2 = \frac{4}{49}:$$

then $v = -m$, or $-\dfrac{3}{7}\,m$.

The first value of v gives, for the first lens, a plano-convex lens, its plane side towards the object-glass: the second gives

$$r = \frac{7}{4}\,m, \quad s = \frac{7}{10}\,m.$$

The second lens, also, must have one of these forms, but must be

placed in the opposite position. In all these combinations,

$$\Sigma(U) = \Sigma(\Upsilon) = \frac{4}{3M}.$$

3d. If the first lens be plano-convex, its plane side towards the object-glass,

$$v = -m, \quad \text{and} \quad V = -\frac{21}{84}m.$$

If equi-convex, $V=0$. If plano-convex in the other position,

$$V = \frac{119}{84}m.$$

4th. If the second lens be plano-convex, its plane side towards the first, $V' = \frac{161}{84}m$. If equi-convex, $V' = \frac{42}{84}m$. If plano-convex in the other position, $V' = \frac{21}{84}m$.

5th. The best combination of common lenses is, two plano-convex lenses with their convexities together. This combination has been considered. The next is, if either of the lenses be equi-convex, and the other plano-convex, with its convexity towards it: this gives

$$\Sigma(V) = \frac{21}{84}m = \frac{m}{4}, \quad \text{and} \quad \Sigma(U) = \frac{19}{12M}, \quad \Sigma(\Upsilon) = \frac{25}{12M}.$$

6th. A single lens to produce the same effect, must have a focal length $\frac{M}{2}$. Its smallest values of U and Υ would be respectively $\frac{14}{3M}$, and $\frac{34}{3M}$.

Ex. 3. Let the eye-piece be the Huyghenian eye-piece.

Here $C=3M$, $B'=-M$, $C'=\frac{M}{2}$, $e=-\frac{m}{6}$, $e'=-\frac{3m}{2}$;

$$A = -\frac{3M}{2},\ \Xi = M,\ A' = M;\ g = -\frac{5m}{6},\ g' = \frac{m}{2}.$$

$$\text{Hence } V = \frac{7}{4m}\left(v - \frac{5}{7}m\right)^2 - \frac{85}{252}m;$$

$$V' = \frac{7}{12m}\left(v' - \frac{5}{7}m\right)^2 + \frac{17}{84}m;$$

$$\Sigma(V) = \frac{7}{4m}\left(v - \frac{5}{7}m\right)^2 + \frac{7}{12m}\left(v' - \frac{5}{7}m\right)^2 - \frac{17}{126}m;$$

$$\Sigma\left(\frac{f}{n}\right) = \frac{8}{9}m.$$

1st. The equation

$$\Sigma(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0$$

is impossible. The nearest approach to it is found by making

$$v = \frac{5}{7}m,\quad v' = \frac{5}{7}m;$$

whence $r = \frac{21}{22}M,\ \ s = -\frac{21}{8}M,\ \ r' = \frac{7}{12}M,\ \ s' = \frac{7}{2}M,$

$$\Sigma(V) = -\frac{17}{126}m,\quad \Sigma(U) = \frac{95}{126M},\quad \Sigma(\Upsilon) = \frac{61}{126M}.$$

2d. The equation $\Sigma(V) = 0$ may be satisfied in an infinite number of ways: and all the forms thus found will give

$$\Sigma(U) = \Sigma(\Upsilon) = \frac{8}{9M}.$$

3d. If the first lens be plano-convex, the plane side towards the object-glass, $V = \frac{19}{12}m$. If equi-convex, $V = \frac{5}{9}m$. If plano-convex in the other position, $V = -\frac{m}{12}$.

4th. If the second lens be plano-convex, the plane side towards the first, $V' = \frac{161}{84} m$. If equi-convex, $V' = \frac{m}{2}$. If plano-convex in the opposite position, $V' = \frac{m}{4}$.

5th. The best combination of common lenses is, both plano-convex, their plane sides towards the eye. This gives

$$\Sigma(V) = \frac{m}{6}, \quad \Sigma(U) = \frac{19}{18\,M}, \quad \Sigma(\Upsilon) = \frac{25}{18\,M}.$$

The next is, the first plano-convex, its plane side towards the eye, and the second equi-convex, it gives

$$\Sigma(V) = \frac{5}{12} m; \quad \Sigma(U) = \frac{47}{36\,M}; \quad \Sigma(\Upsilon) = \frac{77}{36\,M}.$$

6th. A single lens to produce the same effect must have the focal length $\frac{3\,M}{2}$; and, therefore, its smallest values of U and Υ would be $\frac{14}{9\,M}$, and $\frac{34}{9\,M}$.

Ex. 4. Let the eye-piece be the common positive eye-piece, consisting of two lenses of focal length $3\,M$, placed at the distance $2\,M$.

$$\text{Here } C = 3\,M, \quad B' = -M; \quad e = -\frac{m}{6}, \quad e' = -\frac{7\,m}{6};$$

$$Z = -M, \quad A' = 3\,M; \quad g = \frac{7\,m}{6}, \quad g' = \frac{m}{6}.$$

$$\text{Hence, } V = \frac{7}{16\,m}\left(v + \frac{5}{7} m\right)^2 - \frac{295}{1008} m,$$

$$V' = \frac{7}{16\,m}\left(v' - \frac{5}{7} m\right)^2 - \frac{43}{1008} m:$$

$$\Sigma(V) = \frac{7}{16m}\left(v + \frac{5}{7}m\right)^2 + \frac{7}{16m}\left(v' - \frac{5}{7}m\right)^2 - \frac{169}{504}m.$$

And $\Sigma\left(\frac{f}{n}\right) = \frac{4}{9}m = \frac{224}{504}m.$

1st. The equation

$$\Sigma(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0$$

becomes here

$$\frac{7}{16m}\left(v + \frac{5}{7}m\right)^2 + \frac{7}{16m}\left(v' - \frac{5}{7}m\right)^2 - \frac{19}{168}m = 0.$$

This may be satisfied in an infinite number of ways. Indeed, we may give to $\Sigma(V)$ any value not less than

$$-\frac{169}{504}m, \quad \text{or} \quad -\frac{169}{224}\Sigma\left(\frac{f}{n}\right), \quad \text{or} \quad -\frac{3}{4}\Sigma\left(\frac{f}{n}\right) \text{ nearly,}$$

and may, therefore, give to the area of diffusion any of the forms represented in Fig. 6, beginning at the fifth, and continued indefinitely. Of these (as before) we prefer that corresponding to

$$\Sigma(V) = -\tfrac{1}{2}\Sigma\left(\frac{f}{n}\right),$$

or to some smaller negative value. Suppose, then,

$$\Sigma(V) = -\frac{221}{1008}m.$$

This gives $\left(v + \frac{5}{7}m\right)^2 + \left(v' - \frac{5}{7}m\right)^2 = \frac{13}{49}m^2.$

This may be satisfied (for example) by assuming

$$v + \frac{5}{7}m = \frac{3}{7}m, \quad v' - \frac{5}{7}m = -\frac{2}{7}m:$$

then $r = 21\,M$, $s = \frac{21}{13}M$, $r' = \frac{21}{16}M$, $s' = -\frac{21}{2}M$:

and $\Sigma(U) = \frac{227}{1008\,M}$, $\Sigma(\Upsilon) = -\frac{215}{1008\,M}$.

We shall not stop to find forms corresponding to the equation

$$\Sigma(V) = 0.$$

2d. If the first lens be plano-convex, its plane side towards the object-glass, $V = -\frac{231}{1008}m$. If equi-convex, $V = -\frac{70}{1008}m$. If plano-convex in the other position, $V = \frac{189}{1008}m$.

3d. If the second lens be plano-convex, its plane side toward the first, $V' = \frac{441}{1008}m$. If equi-convex, $V' = \frac{182}{1008}m$. If plano-convex, the convexity toward the first, $V' = \frac{21}{1008}m$.

4th. The best combination of common lenses is, two plano-convex lenses with the convexity of each turned towards the other: they give

$$\Sigma(V) = -\frac{105}{504}m, \quad \Sigma(U) = \frac{119}{504\,M}, \quad \Sigma(\Upsilon) = -\frac{91}{504\,M}.$$

The next is, either lens equi-convex, and the other plano-convex, with its convexity towards the former: in this

$$\Sigma(V) = -\frac{49}{1008}m, \quad \Sigma(U) = \frac{399}{1008\,M}, \quad \Sigma(\Upsilon) = \frac{301}{1008\,M}.$$

5th. If an eye-piece of this construction, whatever be the curvatures of the surfaces, be turned end for end, the values of $\Sigma(U)$ and $\Sigma(\Upsilon)$ will not be altered.

6th. A single lens to produce the same effect must have its focal length $= \frac{9M}{4}$, and, therefore, its smallest values of U and Υ would be $\frac{28}{27M}$, and $\frac{68}{27M}$.

Ex. 5. The eye-piece is the old erecting eye-piece.

Here $A=F$, and we must, for the first lens, resort to the investigation which supplies the failure of the general expression. After refraction at the first lens, the rays in a perpendicular plane converge to a line at the distance $\frac{F^2}{U\frac{\beta^2}{2}} = z$: they are incident, therefore, on the next, converging to the distance z, nearly, or $a' = -z$, nearly: therefore,

$$z' = \frac{1}{\frac{1}{F'} - \frac{1}{a'}} - U'\frac{\beta''^2}{2} = \frac{1}{\frac{1}{F'} + \frac{1}{z}} - U'.\frac{\beta''^2}{2} = F' - \frac{F'^2}{z} - U'\frac{\beta''^2}{2}$$

$$= F' - \frac{F'^2}{F^2}\,U\frac{\beta^2}{2} - U'\frac{\beta''^2}{2} = F' - (U+U')\frac{\beta''^2}{2},$$

$$\text{and, therefore, } a'' - A'' = Z' - z' = (U+U')\frac{\beta''^2}{2},$$

the same form as in the other cases. We have, therefore, as in the other examples, to find the values of $\Sigma(U)$ and $\Sigma(\Upsilon)$. Now,

$$e = -\frac{m}{2},\quad e' = \frac{m}{2},\quad e'' = -\frac{m}{2};\quad g = \frac{m}{2},\quad g' = -\frac{m}{2},\quad g'' = \frac{m}{2};$$

$$\text{whence } \Sigma(V) = \frac{7}{3m}(v^2 + v'^2 + v''^2) + 5m.$$

1st. Neither of the equations

$$\Sigma(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0,\quad \Sigma(V) = 0,$$

is possible here.

2d. If we make $\Sigma(V)$ a minimum, we find that all the lenses must be equi-convex. This gives $\Sigma(V)=5m$, whence

$$\Sigma(U)=\frac{7}{M},\quad \Sigma(\Upsilon)=\frac{17}{M}.$$

3d. If either of the lenses be plano-convex, in either position, $\Sigma(V)$ is increased by $\frac{7}{3}m$, $\Sigma(U)$ by $\frac{7}{3M}$, and $\Sigma(\Upsilon)$ by $\frac{7}{M}$.

4th. A single lens, to give the same power, must have the focal length M, and, therefore, its smallest values of U and Υ are $\frac{7}{3M}$ and $\frac{17}{3M}$.

Ex. 6. The eye-piece is the four-glass eye-piece before described.

Here, (as before)

$$e=-\frac{m}{6},\quad e'=\frac{7m}{8},\quad e''=\frac{m}{88},\quad e'''=-m\times{,}439.$$

And, by tracing backwards the positions of the images,

$$A'''=3M,\quad Z''=M\times 2{,}13,\quad A''=-M\times 4{,}556,$$
$$Z'=M\times 10{,}556,\quad A'=M\times 6{,}4404,\quad Z=-M\times 2{,}4404;$$

whence, $g=m\times{,}5764$, $g'=m\times{,}0303$,

$$g''=-m\times{,}3445,\quad g'''=m\times{,}1667.$$

Then $V=\dfrac{1{,}4085}{m}(v+m\times{,}2929)^2-m\times{,}1466;$

$$V'=\frac{{,}8175}{m}(v'+m\times{,}6467)^2+m\times{,}0065;$$

$$V''=\frac{4{,}6041}{m}(v''-m\times{,}2379)^2-m\times{,}0133;$$

$$V'''=\frac{2{,}1205}{m}(v'''-m\times{,}1945)^2+m\times{,}1002:$$

$$\Sigma\,(V) = \frac{1{,}4085}{m}\,(v+m\times{,}2929)^2 + \frac{{,}8175}{m}\,(v'+m\times{,}6467)^2$$

$$+\ \frac{4{,}6041}{m}\,(v''-m\times{,}2379)^2 + \frac{2{,}1205}{m}\,(v'''-m\times{,}1945)^2 - m\times{,}0532.$$

Also $\Sigma\left(\frac{f}{n}\right) = \frac{7}{9}\,m = m\times{,}7778$: $\frac{1}{2}\,\Sigma\left(\frac{f}{n}\right) = m\times{,}3889$.

1st. The equation

$$\Sigma\,(V) + \tfrac{1}{2}\,\Sigma\left(\frac{f}{n}\right) = 0$$

cannot be satisfied. The nearest approach to it is found by making each of the brackets equal to nothing. Then

$r = M\times 24{,}75,\quad s = M\times 1{,}597;\quad r' = -M\times 2{,}521,\quad s' = M\times 1{,}115;$

$r'' = M\times 2{,}050,\quad s'' = M\times 82{,}6;\quad r''' = M\times 1{,}895,\quad s''' = M\times 7{,}204;$

$\Sigma\,(V) = -m\times{,}0532$, whence $\Sigma\,(U) = \frac{{,}7246}{M}$, $\Sigma\,(\Upsilon) = \frac{{,}6182}{M}$.

And this would be the most advantageous combination.

2d. If the first lens be plano-convex, the plane side towards the object-glass, $V = -m\times{,}1444$. If equi-convex, $V = -m\times{,}0256$. If plano-convex in the other position, $V = m\times{,}4056$.

3d. If the second lens be plano-convex, the plane side toward the first, $V' = m\times{,}1351$. If equi-convex, $V' = m\times{,}3485$. If plano-convex, its convexity toward the first, $V' = m\times{,}6638$.

4th. If the third lens be plano-convex, the plane side toward the second, $V'' = m\times 1{,}0824$. If equi-convex, $V'' = m\times{,}2518$. If plano-convex in the opposite position, $V'' = -m\times{,}0128$.

5th. If the fourth lens be plano-convex, the plane side toward the third, $V''' = m\times{,}6910$. If equi-convex, $V''' = m\times{,}1806$. If plano-convex in the other position, $V''' = m\times{,}1411$.

6th. The most advantageous combination, therefore, of common lenses is, all plano-convex, the first and second with their plane sides towards the object-glass, and the third and fourth with their convexities the same way. With this, $\Sigma(V) = m \times ,1190$, whence

$$\Sigma(U) = \frac{,8968}{M}, \quad \Sigma(\Upsilon) = \frac{1,1348}{M}.$$

7th. The eye-piece would be much improved by giving a slight curvature to that side of the fourth lens next the eye, and by making the second lens a meniscus, in which the radius of the concave surface next the first lens is more than double that of the other.

8th. A single lens, for the same power, must have the focal length $M \times 2,16$, and, therefore, its smallest values of U and Υ would be $\frac{1,08}{M}$, and $\frac{2,62}{M}$.

The instances which we have used as examples to our formulæ are taken from the eye-pieces which are, or have been, commonly in use. And the numerical results which we have obtained enable us to judge of their comparative merits, and of the improvements which have been introduced in this branch of telescope-making. Of these, the most remarkable is that of the positive eye-piece. In the form which we have given as the best, two plano-convex lenses, with their convexities towards each other, (and which is, in fact, the common construction), the ratio of its value of $\Sigma(U)$ to that for a single lens is about 1 : 5; whereas, if the lenses be placed in contact, it is 4 : 15; in the Huygenian eye-piece, it is 15 : 31; in the old three-glass eye-piece, 3 : 1, and in the four-glass eye-piece, 2 : 3. The chance, or the reasoning, which led to this construction, avowedly for the purpose of making the

image distinct as far as spherical aberration was concerned, may be considered one of the happiest in the history of optical instruments.

Another improvement, of which the utility is more extensively felt, is the substitution of the four-glass eye-piece for that with three glasses only. With the old three-glass eye-piece we have found that at a given distance from the center of the field, the confusion in the image was three times as great as if a single eye-glass were used. In consequence, the field of view was narrowed in these eye-pieces to a most inconvenient degree. In the four-glass eye-piece, the confusion is not by any means so great as if a single eye-glass were used; and is not one-fourth of that in the three-glass eye-piece. The field of view may, therefore, be made at least twice as broad, preserving the same degree of distinctness; a comfort which every practical observer can easily appreciate.

But the four-glass eye-piece possesses another advantage: it is achromatic. It is true that the three-glass eye-piece (*Cambridge Transactions*, Vol. II. p. 246.) might be made achromatic by increasing the distance between the second and third lenses: but this, I believe, was never done. And the mention of this brings me to another question. How far is the positive eye-piece preferable for the application of a micrometer, to the four-glass eye-piece?

We have seen that the positive eye-piece is superior to any other for the removal of spherical aberration. But it is not achromatic, and cannot be made achromatic (*Cambridge Transactions*, Vol. II. p. 244.); and considerable inconvenience is occasioned by this defect. I have seen several micrometer eye-pieces, in which there was a degree of chromatic confusion, that would have been intolerable in a common perspective-glass. When the object is not far from the center of the field, it is much greater

than the confusion produced by spherical aberration. The latter varies as the square of the distance from the center: but the former varies as the distance (as the variation of tan *FHC*, p. 243 of the former Memoir, is proportional to k). And in this confusion there is nothing to point out the center of the coloured line. If an object-glass be not perfectly achromatic, there is always one brilliant point in the coloured circle which can be fixed on as the image of a star: but if an eye-piece be not achromatic, a star appears a coloured line, with no difference of intensity of light, except that depending on the nature of the colours. It appears to me, therefore, that the first thing to be aimed at for the micrometer eye-piece (supposing the first image formed before the rays enter the eye-piece), is the removal of colour. The only eye-piece now in use which will secure this, is that with four glasses. Its spherical aberration would be three times as great as that of the positive eye-piece, and there would be the loss of light resulting from two additional glasses, but I believe that these defects would be well compensated by the removal of the coloured fringes. The field-glass ought to be plano-convex; the first and fourth, crossed lenses with their more convex sides turned toward each other, and the third, a deep meniscus with its convexity towards the field-glass.

I shall make one more remark, suggested by the expression for the extent of the confusion. It appears that the impossibility of making telescopes perfect depends on this circumstance; that the sum of the powers of the lenses is positive. Now, would it be possible to construct an eye-piece, satisfying the conditions of achromatism, &c., in which some of the lenses were concave?*

* There is another reason which makes it probable that concave lenses might be advantageously introduced. The value of V contains an arbitrary square (depending on ε) whose

I do not know of any form which could be conveniently used, and only mention this as one of the objects to which, in the ulterior improvements of telescopes, the attention of artists might be properly directed.

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whose coefficient is $\frac{n+2}{n} \cdot \frac{f}{(e-g)^2}$. When one lens is concave, or f is negative, we can give to this term a negative value as large as we please, and, therefore, we can make $\Sigma(V)$ as small as we please, and, consequently, the equation

$$\Sigma(V) + \tfrac{1}{2}\Sigma\left(\frac{f}{n}\right) = 0$$

can always be satisfied. By substituting for a convex lens, a convex and a concave lens in contact, whose combined power is the same as that of the convex lens, we may always satisfy this equation, without altering the form or the achromatism, in those eye-pieces where otherwise it is not possible, as in the Huyghenian and four-glass eye-pieces.

G. B. AIRY.

TRINITY COLLEGE,
July 28, 1827.

ADDITION TO THE MEMOIR

ON THE

ACHROMATISM OF EYE-PIECES.

If the object-glass be supposed very distant, the equations for different forms of the achromatic eye-piece (*Cambridge Transactions*, Vol. II. p. 243, &c.) are symmetrical with respect to the first and last lenses, the second and last but one, &c., as well as for the distances. If, then, the eye-piece were turned end for end, the equation would still be true, and, therefore, the eye-piece thus formed would also be achromatic.

If we put t for the tangent of the visual angle, the chromatic variation of this tangent, or δt, expresses the angular extent of the coloured line, which is the apparent image of a point: and if, in this expression, we suppose one of the distances, as b, increased by db, the alteration in the expression will shew the effect on the chromatic dispersion, produced by altering the distance of the lenses. But, as it is impossible (from the nature of the investigation) to determine whether t is to be taken with a positive or negative sign, it will be impossible to say whether the value of $d.\delta.t$ shews that, upon increasing b, the image formed by the most refrangible rays, is brought nearer to, or farther from, the center of the field of view. To avoid this difficulty, divide by t: then, it is plain that if $\frac{d.\delta.t}{t}$ be positive, it shews that, upon increasing b, the image formed by the most refrangible rays is, with respect to that formed by the mean rays, moved from the center of the field; or towards that center, if $\frac{d.\delta.t}{t}$ be negative.

And this applies as well when the equation $\delta t=0$ is satisfied, as in any other case: and we can, therefore, determine with ease the effect of altering one distance of the glasses of an achromatic eye-piece, while the others remain unaltered.

Using, then, the same notation as before, we find

In the eye-piece of two glasses,

$$\frac{d.\delta.t}{t}=-\frac{\delta n}{n-1}.da.\frac{2}{a}.$$

In the three-glass eye-piece,

$$\frac{d.\delta.t}{t}=\frac{\delta n}{n-1}.db.\frac{3a-2.\overline{p+q}}{bp+\overline{a+b}.q+ar-2ab}.$$

In the four-glass eye-piece,

$$\frac{d.\delta.t}{t}=\frac{\delta n}{n-1}.db.\frac{(2.\overline{r+s}-3c)(2.\overline{p+q}-3a)-ac}{ac(q+r-2b)+(bc-rs)(p+q)+(ba-pq)(r+s)}.$$

(The denominators of these fractions have been simplified by means of the equation $\delta t=0$.)

If, upon substitution in these expressions, the fraction is positive, it indicates that an increase of a or b makes the image formed by the blue or most refrangible rays, exterior to that formed by the yellow rays: if the fraction is negative, the yellow image, on increasing a or b, will be exterior to the blue image.

Practical Rules *for the Construction of* Eye-Pieces.

For a single eye-glass.

1. For distinctness, the lens should be equi-convex.

2. If it be rather more convex on the side next the object-glass,

the distinctness will not be much injured, and the distortion will be somewhat diminished.

For a double eye-glass, the two lenses in contact.

1. For distinctness, the lenses should be two similar crossed lenses, the radii of their surfaces as 1 : 6, and the more convex side of each turned towards the other.

2d. The distortion will be diminished by making the first lens (reckoning from the object-glass) equi-convex, and the second meniscus.

For a positive eye-piece.

1. Let the two lenses be plano-convex of focal length 3, placed at the distance 2, with the convex side of each turned towards the other.

2. This distance is somewhat less than that adopted by some of the most eminent opticians. I prefer it because there is a smaller chance of seeing the dust, &c. on the field-glass, which, when the lenses are nearer, happens frequently, especially if the observer be near-sighted.

3. The distance of the field-bar from the first lens must be $\frac{3}{4}$.

For a negative eye-piece.

1. For distinctness, let the first lens be a meniscus lens of focal length 3, the radii of its surfaces as 11 : 4, and its convexity towards the object-glass. Let the second lens be a crossed lens of focal length 1; the radii of its surfaces as 1 : 6, and its more convex side toward the first.

Let the distance of the lenses be 2.

2. If plano-convex lenses be used, there is a considerable loss of distinctness, and no diminution of distortion.

3. The field-bar should be at the distance 1 from the eye-glass.

4. Any other proportion of the focal lengths of the lenses may be

adopted; the distance between the lenses must, in all cases, be half the sum of their focal lengths. The forms of the lenses, without great error, may be the same as those just given.

5. If a bright object appears yellow, or a dark one blue, at the edge farthest from the center of the field, the lenses must be brought a little nearer together.

For a four-glass eye-piece.

1. For distinctness, let the first and last lenses be crossed lenses of focal length 3, the radii of the surfaces being as 1 : 6, and the more convex side of each turned to the other. Let the field-glass be plano-convex of focal length 4, its plane side toward the eye; and the remaining glass a meniscus of focal length 4, the radii of its surfaces as 25 : 11, and its convexity towards the field-glass. Let the distance of the center of the second lens from that of the first be 4: the distance of the field-glass from the second lens 6, and that of the eye-glass from the field-glass 5,13.

2. The distortion in this eye-piece is very small.

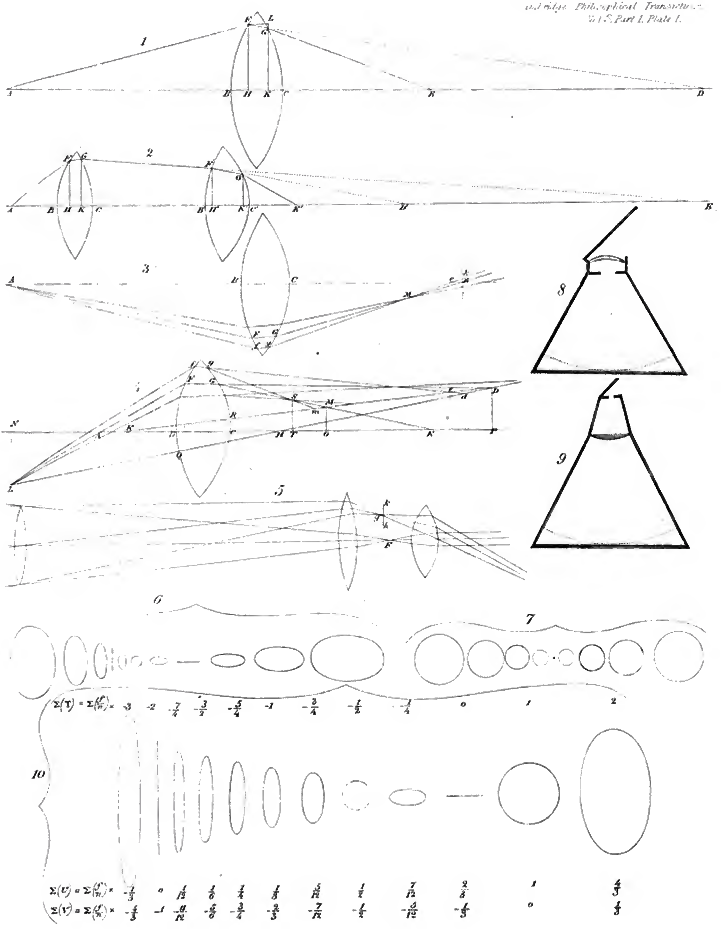
3. The first diaphragm should be at the distance 3 from the first lens. The last diaphragm, or field-bar, at the distance 3 from the eye-glass.

4. Any other focal lengths and distances may be adopted, if the following condition be attended to. Let p be the focal length of the lens next the object-glass; q that of the next lens; r that of the field-glass; and s that of the eye-glass. Let a be the distance between the two first (or the length of the *second pipe*); c the distance between the field-glass and the eye-glass (or the length of the *first pipe*); b the distance between the two middle glasses, (or the distance between the two pipes). Then, p, q, r, s, a, and c, may be taken at pleasure, and then b must be

$$= \frac{qr\,(2\,.\,\overline{a+c}-\overline{p+s}) - (q+r)\,(3\,ac+ps-2\,.\,\overline{cp+as})}{3\,(c\,.\,\overline{p+q}+a\,.\,\overline{r+s}) - 4\,ac - 2\,.\,\overline{p+q}\,.\,\overline{r+s}}\,.$$

The forms of the lenses may be determined, without sensible error, by the rules given above.

5. If a bright object appears yellow, or a dark one blue, at the edge farthest from the center of the field, the two pipes must be pushed a little nearer together.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$\Sigma(T) = \Sigma(\frac{f'}{n}) \times$ -3 -2 $-\frac{7}{4}$ $-\frac{3}{2}$ $-\frac{5}{4}$ -1 $-\frac{3}{4}$ $-\frac{1}{2}$ $-\frac{1}{4}$ 0 1 2
$\Sigma(U) = \Sigma(\frac{f'}{n}) \times$ $-\frac{1}{3}$ 0 $\frac{1}{12}$ $\frac{1}{6}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{5}{12}$ $\frac{1}{2}$ $\frac{7}{12}$ $\frac{2}{3}$ 1 $\frac{4}{3}$
$\Sigma(V) = \Sigma(\frac{f'}{n}) \times$ $-\frac{4}{3}$ -1 $-\frac{11}{12}$ $-\frac{5}{6}$ $-\frac{3}{4}$ $-\frac{2}{3}$ $-\frac{7}{12}$ $-\frac{1}{2}$ $-\frac{5}{12}$ $-\frac{1}{3}$ 0 $\frac{1}{3}$

II. *On Algebraic Notation.*

By THOMAS JARRETT, B.A.

OF CATHARINE HALL.

[Read *Nov.* 12, 1827.]

1. The object of this paper is to propose a notation by means of which the analysis of series may be facilitated, and to suggest a few modifications in the received system of algebraic notation.

2. The principle of the notation about to be proposed, though sufficiently obvious, and at the same time by no means novel, is one capable of being applied much more extensively than it has hitherto been. It is as follows: if a_m is any function of m, and if A represent that operation by which any quantities are combined in a given manner, then the symbol $A^n_m.(a_m)$ may be used to represent the result of such combination applied to n functions, of which the m^{th} is a_m. Again, if v is a function of any quantities, one of which is x, then $B_x.(v)$ may represent the performance on v of any operation in which x alone is considered as variable, and $B^n_x.(v)$ will equally represent the result of that operation when repeated n times.

3. Let then the letter P be taken as an abbreviation of the word *product*, and $P^n_m.a_m$ will denote a product of n factors of which the m^{th} is a_m.

EXAMPLE 1. $x^{2n-1}+1$

$$=(x+1).P_m^{n-1}\left\{\left(x-\cos\frac{2m-1}{2n-1}.\pi-\sqrt{-1}.\sin\frac{2m-1}{2n-1}.\pi\right)\right.$$

$$\left.\left(x-\cos\frac{2m-1}{2n-1}.\pi+\sqrt{-1}.\sin\frac{2m-1}{2n-1}.\pi\right)\right\}$$

$$=(x+1).P_m^{n-1}\left(x^2-2x.\cos\frac{2m-1}{2n-1}.\pi+1\right).$$

Ex. 2. $\cos x=P_m^{\infty}\left(1-\overline{\frac{2x}{2m-1.\pi}}\Big|^2\right)$, and $\sin x=x.P_m^{\infty}\left(1-\overline{\frac{x}{m\pi}}\Big|^2\right)$.

4. PROP. To invert the order of the factors in a given factorial,

$$P^n{}_m.(a_m)=a_1.a_2.a_3\ldots a_n$$

$=a_n.a_{n-1}.a_{n-2}.a_1$, taking the factors in an inverted order

$$=P^n{}_m.a_{n-m+1}.$$

5. PROP. To separate any number of the factors from the rest,

$$P^n{}_m.a_m=(a_1.a_2.a_3\ldots a_r)(a_{r+1}.a_{r+2}\ldots a_n)$$

$$=P^r{}_m.(a_m).P_m^{n-r}.(a_{r+m}).$$

Ex. 1. If $\frac{a_{m+1}}{a_m}=c_m$, m being any positive integer, then shall

$$a_m=a_1.P_r^{m-1}.(c_r).$$

For, $P_r^{m-1}.(c_r)=\frac{P_r^{m-1}(a_{r+1})}{P_r^{m-1}(a_r)}=\frac{P_r^{m-2}(a_{r+1}).a_m}{a_1.P_r^{m-2}.(a_{r+1})}=\frac{a_m}{a_1}$;

$$\therefore\ a_m=a_1.P_r^{m-1}(c_r).$$

Ex. 2. If $\frac{a_{m+2}}{a_m}=c_m$, then $a_{2m-2}=a_0.P_r^{m-1}(c_{2r-2})$,

and $a_{2m-1}=a_1.P_r^{m-1}.(c_{2r-1})$.

$$\text{For, } P_r^{m-1}\,.\,(c_{2r-2}) = \frac{P_r^{m-1}(a_{2r})}{P_r^{m-1}(a_{2r-2})} = \frac{P_r^{m-2}(a_{2r})\,.\,a_{2m-2}}{a_0\,.\,P_r^{m-2}\,.\,(a_{2r})} = \frac{a_{2m-2}}{a_0};$$

$$\therefore\ a_{2m-2} = a_0\,.\,P_r^{m-1}(c_{2r-2}).$$

$$\text{Again, } P_r^{m-1}\,.\,(c_{2r-1}) = \frac{P_r^{m-1}(a_{2r+1})}{P_r^{m-1}(a_{2r-1})} = \frac{P_r^{m-2}(a_{2r+1})\,.\,a_{2m-1}}{a_1\,.\,P_r^{m-2}(a_{2r+1})} = \frac{a_{2m-1}}{a_1};$$

$$\therefore\ a_{2m-1} = a_1\,.\,P_r^{m-1}\,.\,(c_{2r-1}).$$

These two examples will be found of great use in the investigation of the general term of many series.

6. **Prop.** If b is independent of m, then shall

$$P^n{}_m\,(a_m b) = b^n\,.\,P^n{}_m\,(a_m).$$

$$\begin{aligned} \text{For, } P^n{}_m\,.\,(a_m b) &= a_1 b\,.\,a_2 b\,.\,a_3 b \ldots a_n b \\ &= b^n\,.\,a_1\,a_2\,a_3 \ldots a_n \\ &= b^n\,.\,P^n{}_m\,(a_m). \end{aligned}$$

7. **Prop.** Whatever is the form of a_m, $P^0{}_m\,.\,(a_m) = 1$.

For, $P^n{}_m\,.\,(a_m) = P_m^{\overline{n-r}+r}\,.\,(a_m) = P_m^{n-r}\,.\,(a_m)\,.\,P^r{}_m\,(a_{m+n-r})$, (Art. 5.)

$$\therefore\ P_m^{n-r}(a_m) = \frac{P^n{}_m\,(a_m)}{P^r{}_m\,(a_{m+n-r})}.$$

Put now $r = n$, and we get $P^0{}_m\,.\,(a_m) = \dfrac{P^n{}_m\,.\,(a_m)}{P^n{}_m\,.\,(a_m)} = 1.$

8. The most common form in which factorials occur, is that of an arithmetical series. A factorial of this kind consisting of m factors, of which n is the first, and of which the common difference is $\pm\, r$, may be denoted by $\underline{|n}_{\,m,\,\pm r}$; the particular case in which the common difference is -1, we may represent by $\underline{|n}_{\,m}$, and if, in this case, $m = n$, the index subscript may be omitted:

$$\text{Thus } \lfloor n_{\,m,\pm r} = n\,.(n\pm r)\,(n\pm 2r)\ldots(n\pm\overline{m-1}\,.r),$$

$$\lfloor n_{\,m} = n\,.(n-1)\,(n-2)\ldots(n-\overline{m-1}),$$

$$\text{and } \lfloor n = n\,.(n-1)\ldots 1.$$

9. Let the letter S be taken as an abbreviation of the word *sum*, and $S^n_m\,.\,a_m$ will represent the sum of n terms of a series of which the m^{th} term is a_m.

Ex. $\text{Log}_\epsilon\,.\,x = \overset{\infty}{S}_m\,(-1)^{m-1}\,.\,\dfrac{(x-1)^m}{m}.$

10. Prop. To invert a given series,

$$S^n_m\,.\,a_m = a_1 + a_2 + a_3 + \&\text{c.} + a_n$$
$$= a_n + a_{n-1} + a_{n-2} + \&\text{c.} + a_1, \text{ by inverting the series}$$
$$= S^n_m\, a_{n-m+1}\,.$$

11. Prop. To divide a given series into two series.

$$S^n_m\,.\,a_m = (a_1 + a_2 + a_3 + \&\text{c.} + a_r) + (a_{r+1} + a_{r+2} + \&\text{c.} + a_n)$$
$$= S^r_m\,.\,a_m + S^{n-r}_m\,.\,a_{r+m}.$$

12. Prop. To separate the even and odd terms of a series,

$$\overset{\infty}{S}_m\,.\,a_m = a_1 + a_2 + a_3 + \&\text{c.}$$
$$= (a_1 + a_3 + a_5 + \&\text{c.}) + (a_2 + a_4 + a_6 + \&\text{c.})$$
$$= \overset{\infty}{S}_m\,.\,a_{2m-1} + \overset{\infty}{S}_m\, a_{2m}.$$

13. Theorem. If b is independent of m, then shall

$$S^n_m\, b\,a_m = b\,.\,S^n_m\,.\,a_m.$$

$$\text{For, } S^n_m b\,a_m = b\,a_1 + b\,a_2 + b\,a_3 + \&\text{c.} + b\,a_n$$
$$= b\,\{a_1 + a_2 + a_3 + \&\text{c.} + a_n\}$$
$$= b\,.\,S^n_m\,.\,a_m\,.$$

14. The symbol $S^r_m . S^s_n\, a_{m,n}$ will correctly represent the sum of a series consisting of r terms, of which the m^{th} term is the series $S^s_n\, a_{m,n}$; and the same notation may be extended to any number of symbols of summation.

15. THEOREM. $(S^r_m a_m.)\,(S^s_n b_n) = S^r_m a_m . S^s_n b_n.$

For, $(S^r_m a_m)\,(S^s_n b_n) = a_1 . S^s_n b_n + a_2 S^s_n b_n + \text{\&c.} + a_r . S^s_n b_n,$

by actual multiplication,

$$= S^r_m a_m . S^s_n . b_n.$$

16. THEOREM. If r is independent of n, and s of m, then shall

$$S^r_m . S^s_n . a_{m,n} = S^s_n S^r_m . a_{m,n}.$$

For, $S^r_m . S^s_n\, a_{m,n} = S^r_m \{a_{m,1} + a_{m,2} + \text{\&c.} + a_{m,n} + \text{\&c.} + a_{m,s}\}$

$$= S^r_m a_{m,1} + S^r_m a_{m,2} + \text{\&c.} + S^r_m a_{m,n} + \text{\&c.} + S^r_m . a_{m,s}$$

$$= S^s_n . S^r_m a_{m,n}.$$

17. PROP. To arrange $S^n_m . S^m_r . a_{m,r} . x^{r-1}$ according to powers of x.

$$S^n_m . S^m_r . a_{m,r} . x^{r-1} = S^1_r . a_{1,r} . x^{r-1} + S^2_r a_{2,r} . x^{r-1}$$
$$+ S^3_r a_{3,r} . x^{r-1} + \text{\&c.} + S^n_r . a_{n,r} . x^{r-1}.$$

Taking successively the 1st, 2d, and 3d, &c. terms of these different series, we get

$$S^n_m . S^m_r a_{m,r} . x^{r-1} = (a_{1,1} + a_{2,1} + a_{3,1} + \text{\&c.} + a_{n,1})$$
$$+ x\,(a_{2,2} + a_{3,2} + a_{3,2} + \text{\&c.} + a_{n,2})$$
$$+ x^2\,(a_{3,3} + a_{4,3} + a_{5,3} + \text{\&c.} + a_{n,3}) + \text{\&c.} + x^{n-1} . a_{n,n})$$
$$= S^n_r . a_{r,1} + x . S^{n-1}_r a_{r+1,2} + x^2 . S^{n-2}_r a_{r+2,3} + \text{\&c.} + x^{n-1} . a_{n,n}$$
$$= S^n_m . x^{m-1} . S^{n-m+1}_r . a_{r+m-1,m}.$$

18. **Theorem.** $\overset{\infty}{S}_m \overset{\infty}{S}_n a_{m,n} = \overset{\infty}{S}_m S^m{}_n \,.\, a_{m-n+1,n}$.

For, $\overset{\infty}{S}_m \overset{\infty}{S}_n a_{m,n}$ contains every term in which the sum of the indices subscript amount to any positive integer from 2 to ∞, that is, in which it amounts to $m+1$ from $m=1$ to $m=\infty$.

Also $S^m{}_n a_{m-n+1,n}$ contains *every* term in which the sum of the indices subscript amounts to $m+1$.

$\therefore\ \overset{\infty}{S}_m S^m{}_n a_{m-n+1,n}$ contains exactly all the terms of $\overset{\infty}{S}_m \overset{\infty}{S}_n a_{m,n}$.

19. **Theorem.** $\overset{\infty}{S}_m \overset{\infty}{S}_n \overset{\infty}{S}_r a_{m,n,r} = \overset{\infty}{S}_m S^m{}_n \,.\, S^n{}_r \,.\, a_{m-n+1,\,n-r+1,\,r}$.

For $\overset{\infty}{S}_n \overset{\infty}{S}_r a_{m,n,r} = \overset{\infty}{S}_n S^n{}_r a_{m,\,n\ \ r+1,\,r}$, by the last Article;

$$\therefore\ \overset{\infty}{S}_m \overset{\infty}{S}_n \overset{\infty}{S}_r a_{m,n,r} = \overset{\infty}{S}_m \overset{\infty}{S}_n S^n{}_r a_{m,\,n-r+1,\,r}$$

$$= \overset{\infty}{S}_m S^m{}_n S^n{}_r a_{m-n+1,\,n-r+1,\,r},$$

by the same Article.

In precisely the same manner we may show that

$$\overset{\infty}{S}_m \,.\, \overset{\infty}{S}_{m_1} \,.\, \overset{\infty}{S}_{m_2} \ldots \overset{\infty}{S}_{m_s} \,.\, \phi\,(m,\ m_1,\ m_2,\ \&\text{c. } m_s)$$

$$= \overset{\infty}{S}_m \,.\, S_{m_1}{}^m \,.\, S_{m_2}{}^{m_1} \ldots S_{m_s}{}^{m_{s-1}} \,.\, \phi\,(m-m_1+1,\ m_1-m_2+1,\ \&\text{c. } m_{s-1}-m_s+1,\ m_s).$$

20. **Prop.** To arrange, according to powers of x, the series

$$\overset{\infty}{S}_r \,.\, \phi\,(m,r) \,.\, x^{r-1} \,.\, \overset{\infty}{S}_{r_1} \,.\, \phi_1\,(r,r_1) \,.\, x^{r_1-1} \,.\, \overset{\infty}{S}_{r_2} \ldots \overset{\infty}{S}_{r_s} \,.\, \phi_s\,(r_{s-1},\ r_s) \,.\, x^{r_s-1}.$$

For this purpose we must first remove every quantity to the right of all the symbols of summation (Art. 13.); in the next place, we must write $r-r_1+1$ for r, r_1-r_2+1 for r_1, and so on; observing to leave r_s unaltered; and lastly, place over the symbols of summation, beginning at the second, the quantities r, r_1, r_2, &c. (Art. 19.).

We shall thus find

$$\overset{\infty}{S}_r \phi(m,r) \,.\, x^{r-1} \,.\, \overset{\infty}{S}_{r_1} \phi_1(r,r_1) \,.\, x^{r_1-1} \,.\, \overset{\infty}{S}_{r_2} \ldots \overset{\infty}{S}_{r_s} \phi_s(r_{s-1}, r_s) \,.\, x^{r_s-1}$$

$$= \overset{\infty}{S}_r \,.\, S_{r_1}{}^{r} \ldots S_{r_s}{}^{r_s-1} \,.\, \phi(m, r-r_1+1) \,.\, \phi_1(r-r_1+1, r_1-r_2+1) \ldots$$

$$\phi_{s-1}(r_{s-2}-r_{s-1}+1, r_{s-1}-r_s+1) \,.\, \phi_s(r_{s-1}-r_s+1, r_s) \,.\, x^{r-1}$$

$$= \overset{\infty}{S}_r\, x^{r-1} \,.\, S_{r_1}{}^{r} \phi(m, r-r_1+1) \,.\, S_{r_2}{}^{r_1} \phi_1(r-r_1+1, r_1-r_2+1) \,.\, S_{r_3}{}^{r_2} \ldots .$$

$$S_{r_s}{}^{r_{s-1}} \,.\, \phi_{s-1}(r_{s-2}-r_{s-1}+1, r_{s-1}-r_s+1) \,.\, \phi_s(r_{s-1}-r_s+1, r_s),$$

(Art. 13.)

21. The form in which this last result appears, naturally suggests a notation by which, indices being applied to brackets, many very complicated expressions may be reduced to a very simple form. Let then $\overset{n}{\underset{m}{\{}} a_m$ be used as an abbreviation of the expression $\underset{1}{\{} a_1 \,.\, \underset{2}{\{} a_2 \,.\, \underset{3}{\{} a_3 \,.\, \underset{4}{\{} \ldots \underset{n}{\{} a_n$, where a_m is any function of m, and the result of the last Article will become

$$\overset{\infty}{S}_r \,.\, \phi(m,r) \,.\, x^{r-1} \,.\, \overset{s}{\underset{t}{\{}} \overset{\infty}{S}_{r_t} \,.\, \phi_t(r_{t-1}, r_t)\, x^{r_t-1} = \overset{\infty}{S}_r x^{r-1} \,.\, S_{r_1}{}^{r} \phi(m, r-r_1+1) \,.$$

$$\overset{s-2}{\underset{t}{\{}} S_{r_{t+1}}{}^{r_t} \,.\, \phi_t(r_{t-1}-r_t+1, r_t-r_{t+1}+1)$$

$$\underset{s-1}{\{} S_{r_s}{}^{r_{s-1}} \,.\, \phi_{s-1}(r_{s-2}-r_{s-1}+1, r_{s-1}-r_s+1) \,.\, \phi_s(r_{s-1}-r_s+1, r_s).$$

22. PROP. To arrange according to indices subscript of b, the series

$$\overset{\infty}{S}_m S^m{}_n\, a_{m,n} \,.\, b_n, \quad \overset{\infty}{S}_m S^m{}_n\, a_{m,n} \,.\, b_{2n-1}, \quad \text{and} \quad \overset{\infty}{S}_m S^m{}_n\, a_{m,n} \,.\, b_{2n}.$$

$$\overset{\infty}{S}_m S^m{}_n\, a_{m,n} \,.\, b_n = b_1 \,.\, \overset{\infty}{S}_m a_{m,1} + b_2 \,.\, \overset{\infty}{S}_m a_{m+1,2} + b_3 \,.\, \overset{\infty}{S}_m a_{m+2,3} + \&c.$$

$$= \overset{\infty}{S}_n b_n \,.\, \overset{\infty}{S}_m a_{m+n-1,n}.$$

$$\overset{\infty}{S}_m S^m{}_n a_{m,n} \,.\, b_{2n-1} = b_1 \,.\, \overset{\infty}{S}_m a_{m,1} + b_3 \,.\, \overset{\infty}{S}_m a_{m+1,2} + b_5 \,.\, \overset{\infty}{S}_m a_{m+2,3} + \&c.$$

$$= \overset{\infty}{S}_n b_{2n-1} \,.\, \overset{\infty}{S}_m a_{m+n-1,n},$$

and $$\overset{\infty}{S}_m S^m{}_n a_{m,n} \,.\, b_{2n} = b_2 \,.\, \overset{\infty}{S}_m a_{m,1} + b_4 \,.\, \overset{\infty}{S}_{m+1,2} + b_6 \,.\, \overset{\infty}{S}_m a_{n+2,3} + \&c.$$

$$= \overset{\infty}{S}_n b_{2n} \,.\, \overset{\infty}{S}_m a_{m+n-1,n}.$$

23. We frequently need a symbol to denote the sum of all the combinations that can be formed of n quantities taken m at a time. Let then the letter C be taken as an abbreviation of the word *combinations*, and $C_r^{m,n}(a_r)$ may be used to denote the sum of every possible combination of n quantities, of which the r^{th} is a_r, and of which we are to take m at a time.

24. **Theorem.** If b is independent of r, then shall

$$C_r^{m,n}(a_r b) = b^m \,.\, C_r^{m,n}(a_r).$$

For $C_r^{m,n}(a_r b)$ denotes the sum of a series, each term of which is a product of m quantities, and into each of which quantities b enters as a multiplier; and $C_r^{m,n}(a_r)$ denotes the sum of a series, each term of which is a product of the same m quantities deprived of their multiplier b, and hence, by Art. 6 and 13, the truth of the proposition is obvious.

25. **Theorem.** $$\frac{C_r^{m,n}(a_r)}{P_r^n(a_r)} = C_r^{n-m,n} \,.\, \left(\frac{1}{a_r}\right).$$

For, the numerator of the left side of this equation consists of every possible combination of n quantities taken m at a time; and hence that side of the equation consists of a series of fractions, in which the numerator of each is unity, and in which the denominators are formed by taking away, in every possible manner, m of n given quantities, and will therefore consist of every combination of those n quantities, taken $m-n$ at a time.

26. The above symbol will be found extremely useful in the theory of eliminations; but the following examples may be taken as specimens of a more general application.

Ex. 1. $P_r^n(x+a_r) = S_m^n x^{m-1} \,.\, C_r^{n-m+1,n}(a_r)$.

Ex. 2. If a_r is the r^{th} root of the equation, $0 = S_m^{n+1} a_{m-1} x^{m-1}$, then will $a_{m-1} = C_r^{n-m+1,n}(-a_r)$.

Ex. 3. To transform the equation $0 = S_m^{n+1} a_{m-1} x^{m-1}$, of which the r^{th} root is a_r into one of which the r^{th} root is ca_r.

The equation sought will be

$$\begin{aligned} 0 &= S_m^{n+1} x^{m-1} \,.\, C_r^{n-m+1,n}(-ca_r) \\ &= S_m^{n+1} x^{m-1} \,.\, c^{n-m+1} \,.\, C_r^{n-m+1,n}(-a_r) \quad \text{(Art. 24.)} \\ &= S_m^{n+1} a_{m-1} \,.\, c^{n-m+1} \,.\, x^{m-1}. \end{aligned}$$

Ex. 4. If in the equation

$$0 = S_m^{n+1} a_{m-1} x^{m-1},$$

we have

$$a_{m-1} = a_{n-m+1},$$

then if a_r is a root, a_r^{-1} shall also be a root.

$$\begin{aligned} \text{For, } 0 &= S_m^{n+1} a_{m-1} x^{m-1} \\ &= S_m^{n+1} a_{n-m+1} x^{m-1}, \text{ by hypothesis,} \\ &= S_m^{n+1} x^{m-1} \,.\, C_r^{m-1,n}(-a_r); \end{aligned}$$

and, dividing by $P_r^n(-a_r)$,

$$\begin{aligned} 0 &= S_m^{n+1} x^{m-1} \,.\, \frac{C_r^{m-1,n}(-a_r)}{P^n{}_r(-a_r)} \\ &= S_m^{n+1} x^{m-1} \,.\, C_r^{n-m+1,n}(-a_r^{-1}) \quad \text{(Art. 25.)}, \end{aligned}$$

which is an equation of which the r^{th} root is a_r^{-1}.

27. The letter D being considered as an abbreviation of the word *difference*, the symbol $D^n_x . u$ will denote the n^{th} difference of u, x being the independent variable, and the increment of x being arbitrary. The case in which Dx, or the difference of x, is unity being of very common occurrence, we may denote by $\Delta_x^n . u$, the n^{th} difference taken on that supposition *.

28. We sometimes need a symbol to denote the new value which a function of any variable assumes on giving to that variable an arbitrary increment. Arbogast has proposed to use the letter E (*Etat*), and this may be adopted with advantage provided we add the index subscript, to denote the independent variable: thus, $E_x . (u)$ will denote that $x + Dx$ has been substituted for x in u; and $E^n_x (u)$ will signify that this operation has been performed n times.

29. By a slight modification of a notation proposed by Euler we may greatly abridge the method of expressing partial differential coefficients. Instead of $\frac{d^{m+n} u}{dx^m . dy^n}$ he proposed to write $\frac{d^m}{x} . \frac{d^n}{y} . u$; if we change this last into $d^m_x d^n_y . u$ we get an expression much more simple. Should the quantity to be differentiated be a power of the independent variable itself, the index subscript may be omitted; thus $d . x^n$ may be written for $d_x . x^n$.

30. The n^{th} differential coefficient appears most frequently with $\underline{|n}$ as a divisor; we may therefore abridge our notation still farther by writing $d^n_x : u$ for $\frac{d^n_x u}{\underline{|n}}$.

* Prony, among others, has used the index subscript to denote the independent variable; but no notation has hitherto been adopted to distinguish the case in which Dx is arbitrary from that in which it is unity.

31. It is frequently necessary to give a particular value to the independent variable after the differentiation has been performed. This may be denoted by placing the particular value of the independent variable, as an index subscript, to the right of that which denotes the general value of independent variables. Thus $d^n{}_{x,a}.u$ signifies that, after taking the n^{th} differential coefficient with respect to x, we must put $x = a$.

32. The same notation being extended to integrals, $\int_{x,b}u - \int_{x,a}u$, or, as it may be still more conveniently expressed, $(\int_{x,b} - \int_{x,a})\,u$ will denote the integral of u taken with respect to x, between the limits a and b. In the same manner we shall have

$$S^n{}_m\, a_m = (\Delta_n{}^{-1} - \Delta_{n,o}{}^{-1})\, a_{n+1}.$$

33. The following examples will best show the application and utility of our notation.

Ex. 1. $$x^n = 1 + S^t{}_s \underline{\lfloor n}_{s} . \frac{(x-1)^s}{\lfloor s} + (x-1)^{t+1} . S_s{}^{n-t} \underline{\lfloor n-s}_{t} . \frac{x^{s-1}}{\lfloor t};$$

n being a positive integer*.

$$\text{For,}\quad \frac{x^n - 1}{x - 1} = S^n{}_r x^{r-1}, \text{ by division,}$$

$$\text{and}\quad x^n = 1 + (x-1) . S^n{}_r x^{r-1},$$

by multiplication and transposition.

Whence, by successive substitution,

$$x^n = 1 + (x-1) . S^n{}_r \{1 + (x-1) . S_{r_1}{}^{r-1}$$
$$\underset{1}{\{} 1 + (x-1) . S_{r_2}{}^{r_1-1} \underset{2}{\{} \ldots \underset{t-1}{\{} 1 + (x-1) . S_r{}^{r_{t-1}-1} x^{r_t-1} \underset{t-1}{\}} \ldots \}$$

* The first four examples are taken from "The true developement of the Binomial Theorem," by Mr. Swinburne and the Rev. T. Tylecote, of St. John's College; a work which contains the only rigid proof of that theorem that has yet appeared.

$$= 1 + (x-1)\, S^n{}_r \,.\, 1 + (x-1)^2 \,.\, S^n{}_r S_{r_1}{}^{r-1} \,.\, 1 + \&c.$$

$$+ (x-1)^t \,.\, S^n{}_r S_{r_1}{}^{r-1} \ldots S_{r_{t-1}}{}^{r_{t-1}-1} \,.\, 1 + (x-1)^{t+1} \,.\, S_r{}^n S_{r_1}{}^{r-1} \ldots S_{r_2}{}^{r_{t-1}-1} x^{r_t-1}$$

$$= 1 + S^t{}_s (x-1)^s \,.\, S^n{}_r S_{r_1}{}^{r-1} \ldots S_{r_{s-1}}{}^{r_{s-2}-1} \,.\, 1$$

$$+ (x-1)^{t+1} \,.\, S^n{}_r S_{r_1}{}^{r-1} \ldots S_{r_t}{}^{r_{t-1}-1} x^{r_t-1}$$

$$= 1 + S^t{}_s (x-1)^s \,.\, S_r{}^{n+1-s} \,.\, S_{r_1}{}^r \ldots S_{r_{s-1}}{}^{r_{s-2}} \,.\, 1$$

$$+ (x-1)^{t+1} \,.\, S_r{}^{n-t} S_{r_1}{}^r \ldots S_{r_t}{}^{r_{t-1}} x^{r_t-1} \quad \text{(Art. 19.)} \ldots .(1).$$

Now $S_{r_1}{}^r S_{r_2}{}^{r_1} \,.\, S_{r_3}{}^{r_2} \,.\, 1 = S_{r_1}{}^r S_{r_2}{}^{r_1} \,.\, r_2 = S_{r_1}{}^r \Delta_{r_1}{}^{-1} (r_1+1)$

$$= S_{r_1}{}^r \lfloor\underline{r_1+1}_{\,2} \,.\, \frac{1}{\lfloor\underline{2}} = \Delta_r{}^{-1} \,.\, \lfloor\underline{r+2}_{\,2} \,.\, \frac{1}{\lfloor\underline{2}} = \lfloor\underline{r+2}_{\,3} \,.\, \frac{1}{\lfloor\underline{3}};$$

and similarly, $S_{r_1}{}^r \ldots S_{r_{s-1}}{}^{r_{s-2}} \,.\, 1 = \lfloor\underline{r+s-2}_{\,s-1} \,.\, \dfrac{1}{\lfloor\underline{s-1}}$.

Whence $S_r{}^{n+1-s} \,.\, S_{r_1}{}^r \ldots S_{r_{s-1}}{}^{r_{s-2}} \,.\, 1 = \lfloor\underline{n}_{\,s} \,.\, \dfrac{1}{\lfloor\underline{s}}$.

Also $S_{r_1}{}^r S_{r_2}{}^{r_1} S_{r_3}{}^{r_2} \,.\, x^{r_3-1} = S_{r_1}{}^r S_{r_2}{}^{r_1} x^{r_2-1} \,.\, S_{r_3}{}^{r_1+1-r_2} \,.\, 1$

$$= S_{r_1}{}^r S_{r_2}{}^{r_1} \,.\, x^{r_2-1} \,.\, (r_1+1-r_2), \quad \text{(Art. 16)}$$

$$= S_{r_1}{}^r x^{r_1-1} \,.\, S_{r_2}{}^{r+1-r_1} (r_1+r_2-1+1-r_1)$$

$$= S_{r_1}{}^r x^{r_1-1} \,.\, S_{r_2}{}^{r+1-r_1} \,.\, r_2 = S_{r_1}{}^r x^{r_1-1} \,.\, \Delta_r{}^{-1} \,.\, (r+2-r_1)$$

$$= S_{r_1}{}^r x^{r_1-1} \,.\, \lfloor\underline{r+2-r_1}_{\,2} \,.\, \frac{1}{\lfloor\underline{2}};$$

and similarly, $S_{r_1}{}^r \ldots S_{r_t}{}^{r_{t-1}} x^{r_t-1} = S_{r_t}{}^r x^{r_t-1} \,.\, \lfloor\underline{r+t-1-r_1}_{\,t-1} \,.\, \dfrac{1}{\lfloor\underline{t-1}}$;

whence $S_r{}^{n-t} \,.\, S_{r_1}{}^r \ldots S_{r_t}{}^{r_{t-1}} \,.\, x^{r_t-1} = S_r{}^{n-t} x^{r-1} \,.\, \lfloor\underline{n-r}_{\,t} \,.\, \dfrac{1}{\lfloor\underline{t}}$;

and, substituting in (1), we get

$$x^n = 1 + S^t_s \underline{|n}_{\,s} \frac{(x-1)^s}{\underline{|s}} + (x-1)^{t+1} . S_s^{n-t} \underline{|n-s}_{\,t} . \frac{x^{s-1}}{\underline{|t}}.$$

$$= 1 + S^t_s \underline{|n}_{\,s} . \frac{(x-1)^s}{\underline{|s}} + (x-1)^{t+1} . S_s^{n-t} \underline{|t+1-s}_{\,t} . \frac{x^{n-t-s}}{\underline{|t}}$$

Ex. 2. $$x^{-n} = 1 + S^t_s \underline{|-n}_{\,s} . \frac{(x-1)^s}{\underline{|s}} + (1-x)^{t+1} . S^n_s \underline{|n+t-s}_{\,t} . \frac{x^{-s}}{\underline{|t}}.$$

For, $\dfrac{x^{-n}-1}{x^{-1}-1} = S^n_r x^{-(r-1)}$,

$$x^{-n} = 1 + (x^{-1}-1)\, S^n_r x^{-r+1},$$

by multiplication and transposition,

$$= 1 + (1-x)\, S^n_r x^{-r}$$

$$= 1 + S^t_s (1-x)^s . S^n_r S_{r_1}^{r} \ldots S_{r_{s-1}}^{r_{s-2}} . 1$$

$$+ (1-x)^{t+1} . S^n_r S_{r_1}^{r} \ldots S_{r_t}^{r_{t-1}} x^{-r_t},$$

by successive substitution.

But $S^n_r S_{r_1}^{r} \ldots S_{r_{s-1}}^{r_{s-2}} . 1 = \underline{|n+s-1}_{\,s} . \dfrac{1}{\underline{|s}}$,

and $S^n_r S_{r_1}^{r} \ldots S_{r_t}^{r_{t-1}} x^{-r_t} = x^{-1} . S^n_r S_r^{r_1} \ldots S_{r_t}^{r_{t-1}} x^{-(r_t-1)}$

$$= x^{-1} S^n_r \underline{|n+t-r}_{\,t} . \frac{x^{-(r-1)}}{\underline{|t}}$$

$$= S^n_r \underline{|n+t-r}_{\,t} . \frac{x^{-r}}{\underline{|t}}.$$

$$\therefore\ x^{-n} = 1 + S^t_s \underline{|n+s-1}_{\,s} . \frac{(1-x)^s}{\underline{|s}} + (1-x)^{t+1} . S^n_s \underline{|n+t-s}_{\,t} . \frac{x^{-s}}{\underline{|t}}.$$

But $\underline{\lfloor n+s-1}_{s} = (-1)^{s} \underline{\lfloor -n-s+1}_{s,1} = (-1)^{s} . \underline{\lfloor -n}_{s}$;

$$\therefore\ x^{-n} = 1 + S^{t}_{s} \underline{\lfloor -n}_{s} . \frac{(x-1)^{s}}{\lfloor s} + (1-x)^{t+1} . S^{n}_{s} \underline{\lfloor n+t-s}_{t} . \frac{x^{-s}}{\lfloor t} .$$

Ex. 3. $$\underline{\lfloor m}_{t+1} - \frac{1}{\lfloor r} . \underline{\lfloor \frac{m}{n}}_{r+1} . S^{r}_{s} \frac{(-1)^{s-1}}{\lfloor s-1} . \underline{\lfloor r}_{s-1} . \underline{\lfloor \overline{r+1-s}.n}_{t+1} . \frac{1}{\frac{m}{n} - \overline{r+1-s}} = 0,$$

$$\left\{\begin{matrix} t = 0 \\ t = r-1 \end{matrix}\right\} :$$

and $$\underline{\lfloor m}_{r+1} - \frac{1}{\lfloor r} . \underline{\lfloor \frac{m}{n}}_{r+1} . S^{r}_{s} \frac{(-1)^{s-1}}{\lfloor s-1} . \underline{\lfloor r}_{s-1} . \underline{\lfloor \overline{r+1-s}.n}_{r+1} . \frac{1}{\frac{m}{n} - \overline{r+1-s}} = n^{r+1} . \underline{\lfloor \frac{m}{n}}_{r+1} .$$

Put $$f(r,t) = \underline{\lfloor m}_{t+1} - \frac{1}{\lfloor r} . \underline{\lfloor \frac{m}{n}}_{r+1} . S^{r}_{s} \frac{(-1)^{s-1}}{\lfloor s-1} . \underline{\lfloor r}_{r-1} . \underline{\lfloor \overline{r+1-s}.n}_{t+1} . \frac{1}{\frac{m}{n} - \overline{r+1-s}} \quad (1),$$

then $$\underline{\lfloor \overline{r+1-s}.n}_{t+1} = \underline{\lfloor \overline{r+1-s}.n}_{t} . (\overline{r+1-s}.n - t)$$

$$= \underline{\lfloor \overline{r+1-s}.n}_{t} \{(nr-t) - n(s-1)\} ;$$

$$\therefore f(r,t) = \underline{\lfloor m}_{t+1} - \frac{1}{\lfloor r} . \underline{\lfloor \frac{m}{n}}_{r+1} \left\{ (nr-t) . S^{r}_{s} \frac{(-1)^{s-1}}{s-1} \underline{\lfloor r}_{s-1} . \frac{\underline{\lfloor \overline{r+1-s}.n}_{t}}{\frac{m}{n} - (r-s+1)} \right.$$

$$\left. - n . S^{r}_{s} \frac{(-1)^{s-1}}{\lfloor s-1} . \underline{\lfloor r}_{s-1} . (s-1) . \underline{\lfloor \overline{r+1-s}.n}_{t} . \frac{1}{\frac{m}{n} - (r+1-s)} \right\} .$$

But $$- n . S^{r}_{s} \frac{(-1)^{s-1}}{\lfloor s-1} . \underline{\lfloor r}_{s-1} . (s-1) . \underline{\lfloor \overline{r+1-s}.n}_{t} . \frac{1}{\frac{m}{n} - (r+1-s)} =$$

$$= o - n\, S_s^{r-1}\, \frac{(-1)^s}{\lfloor s} \,.\, \lfloor r_{s} \,.\, s \,.\, \lfloor \overline{r-s\,.\,n}_{t} \,.\, \frac{1}{\frac{m}{n} - (r-s)} \quad \text{(Art. 11.)}$$

$$= nr \,.\, S_s^{r-1}\, \frac{(-1)^{s-1}}{\lfloor s-1} \,.\, \lfloor r-1_{s-1} \,.\, \lfloor \overline{r-s\,.\,n}_{t} \,.\, \frac{1}{\frac{m}{n} - (r-s)};$$

$$\therefore f(r,t) = \lfloor m_{t+1} + (nr-t)\,.\,f(r,t-1) - \lfloor m_{t}\,.\,(nr-t)$$

$$+ n\,.\left(\frac{m}{n} - r\right).f(r-1,t-1) - n\left(\frac{m}{n} - r\right)\lfloor m_{t}.$$

But $\lfloor m_{t+1} - \lfloor m_{t}.(nr-t) - n\left(\frac{m}{n} - r\right).\, m$

$$= \lfloor m_{t}\, \{m - t - nr + t - m + nr\} = 0;$$

$$\therefore f(r,t) = (nr-t)\,.f(r,t-1) + (m-nr)\,.f(r-1,t-1) \ldots\ldots (2).$$

For t, put $t-1$, then

$$f(r,t-1) = (nr-t+1)f(r,t-2) + (m-nr)f(r-1,t-2).$$

For r and t put $r-1$ and $t-1$ respectively, then

$$f(r-1,t-1) = (n\,.\,\overline{r-1} - t + 1)f(r-1,t-2) + (m - n\,.\,\overline{r-1})f(r-2,t-2)$$

Put $A_{v,s}$ = coefficient of $f(r-v,\ t-s)$ in $f(r,t)$, and we get

$$f(r,t) = A_{0,1}\,.f(r,t-1) + (m-nr)\,.f(r-1,t-1)$$

$$= A_{0,2}\,.f(r,t-2) + A_{1,2}\,.f(r-1,t-2)$$

$$+ (m-nr)\,(m-n\,.\,\overline{r-1})\,.f(r-2,t-2)$$

$$= A_{0,2}\,.f(r,t-2) + A_{1,2}\,.f(r-1,t-2)$$

$$+ n^2\,.\,\lfloor \tfrac{m}{n} - r_{2,1}\,.f(r-2,t-2)$$

$$= S^s{}_v\,.\,A_{v-1,s}\,.f(r+1-v,\ t-s)$$

$$+ n^s\,.\,\lfloor \tfrac{m}{n} - r_{s,1}\,.f(r-s,t-s).$$

Put $s = t - 1$; then

$$f(r, t) = S_v^{t-1} A_{v-1, t-1} . f(r+1-v, 1) + n^{t-1} . \underline{\left|\frac{m}{n} - r\right.}_{\,t-1,1} . f(r+1-t, 1) \ldots (3).$$

Put $t = r$, then

$$f(r, r) = S_v^{r-1} A_{v, r-1} . f(r+1-v, 1) + n^{r-1} . \underline{\left|\frac{m}{n} - r\right.}_{\,r-1,1} . f(1, 1).$$

But $f(1, 1) = \underline{\left|m\right.}_{\,2} - \underline{\left|\frac{m}{n}\right.}_{\,2} . \underline{\left|n\right.}_{\,2} . \dfrac{1}{\frac{m}{n} - 1}$

$$= m.(m-1) - \frac{m}{n}.n.(n-1) = m\{m - 1 - n + 1\} = n^2 . \underline{\left|\frac{m}{n}\right.}_{\,2}.$$

And $n^{r-1} . \underline{\left|\frac{m}{n} - r\right.}_{\,r-1,1} . n^2 . \underline{\left|\frac{m}{n}\right.}_{\,2} = n^{r+1} . \underline{\left|\frac{m}{n}\right.}_{\,r+1}.$

$$\therefore f(r, r) = S_v^{r-1} . A_{v, r-1} . f(r+1-v, 1) + n^{r+1} . \underline{\left|\frac{m}{n}\right.}_{\,r+1} \ldots\ldots (4).$$

Again, $f(r, 1) = (nr - 1) f(r, o) + (m - nr) . f(r-1, o)$,

from (2), (5),

and $f(r, o) = m - \dfrac{1}{\underline{\left|r\right.}} . \underline{\left|\frac{m}{n}\right.}_{\,r+1} . S^r_s \dfrac{(-1)^{s-1}}{\underline{\left|s+1\right.}} . \underline{\left|r\right.}_{\,s-1} . \dfrac{\overline{r+1-s}.n}{\frac{m}{n} - \overline{r+1-s}}$

$$= m + \frac{n^2}{\underline{\left|r\right.}} . \underline{\left|\frac{m}{n}\right.}_{\,r+1} . S^r_s \frac{(-1)^{s-1}}{\underline{\left|s-1\right.}} . \underline{\left|r\right.}_{\,s} . \frac{1}{\overline{r+1-s}.n - m}$$

But $\dfrac{1}{\overline{r+1-s}.n - m} = \dfrac{1}{rn - m} . \dfrac{nr - m}{\overline{r+1-s}.n - m}$

$$= \frac{1}{nr - m} . \left\{1 + \frac{(s-1)n}{\overline{r+1-s}.n - m}\right\};$$

$$\therefore f(r,o) = m + \frac{n^2}{\underline{|r}\,.(nr-m)} \cdot \underline{\left|\frac{m}{n}\right.}_{r+1} . S^r_s \frac{(-1)^{s-1}}{\underline{|s-1}} . \underline{|r}_{s} . \left\{1 + \frac{(s-1)\,n}{r+1-s\,.\,n-m}\right\}$$

$$= m + \frac{n^2}{\underline{|r}\,.(nr-m)} \cdot \underline{\left|\frac{m}{n}\right.}_{r+1} \left\{r\,.(1-1)^{r+1}. + o\right.$$

$$\left. + S_s^{r-1} \frac{(-1)^s}{\underline{|s}} . \underline{|r}_{s+1} . s . \frac{n}{r-s\,.\,n-m}\right\}$$

$$= m - \frac{n^3 r}{\underline{|r}\,.(nr-m)} \cdot \underline{\left|\frac{m}{n}\right.}_{r+1} . S_s^{r-1} \frac{(-1)^{s-1}}{\underline{|s-1}} \underline{|r-1}_{s} . \frac{1}{r-s\,.\,n-m}$$

$$= m + \frac{n^2}{\underline{|r-1}} . \underline{\left|\frac{m}{n}\right.}_{r} . S_s^{r-1} \frac{(-1)^{s-1}}{\underline{|s-1}} . \underline{|r-1}_{s} . \frac{1}{r-s\,.\,n-m}$$

$$= f(r-1, o).$$

But $f(1,o) = m - \underline{\left|\frac{m}{n}\right.}_{2} . n . \dfrac{1}{\frac{m}{n}-1} = 0;\quad \therefore f(r,o) = 0.$

Hence, from (5), $f(r,1) = 0$,

and from (3), $f(r,t) = 0, \quad \begin{Bmatrix} t=0 \\ t=r-1 \end{Bmatrix}$

and from (4), $f(r,r) = n^{r+1} . \underline{\left|\frac{m}{n}\right.}_{r+1}$ Q. E. D.

Cor. Put $m = t$, then

$$S^r_s \frac{(-1)^{s-1}}{\underline{|s-1}} . \underline{|r}_{s-1} . \underline{|\overline{r+1-s}\,.\,n}_{t} = 0, \quad \begin{Bmatrix} t=0 \\ t=r-1 \end{Bmatrix};$$

and put $m = r$, then

$$\underline{|r}_{r+1} + \frac{n}{\underline{|r}} . \underline{\left|\frac{r}{n}\right.}_{r+1} . S^r_s \frac{(-1)^{s-1}}{\underline{|s-1}} . \underline{|r}_{s-1} . \underline{|\overline{r+1-s}\,.\,n}_{r} = n^{r+1} . \underline{\left|\frac{r}{n}\right.}_{r+1},$$

$$\text{or} \quad S^r{}_s \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \lfloor\underline{\overline{r+1-s}.n}_{r} = n^r . \lfloor\underline{r}.$$

Ex. 4. $$a^{\frac{m}{n}} = 1 + S^r{}_t \lfloor\underline{\tfrac{m}{n}}_{t} . \frac{\overline{a-1}|^t}{\lfloor\underline{t}} + \overset{\infty}{S}_t \frac{(a^{\frac{1}{n}}-1)^{r+t}}{\lfloor\underline{r+t}}$$

$$\left\{ \lfloor\underline{m}_{t+r} - \lfloor\underline{\tfrac{m}{n}}_{r+1} . \frac{1}{\lfloor\underline{r}} . S^r{}_s \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \lfloor\underline{\overline{r+1-s}.n}_{t+r} . \frac{1}{\frac{m}{n} - \overline{r+1-s}} \right\}.$$

Put $a^{\frac{1}{n}} - 1 = u$, then $a^{\frac{1}{n}} = 1 + u$, $a = (1+u)^n$,

and $(a-1)^r = (\overline{1+u}|^n - 1)^r = S_s^{r+1} \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . (1+u)^{n.\overline{r+1-s}}$,

by the binomial theorem,

$$= S_s^{r+1} \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \overset{\infty}{S}_t \lfloor\underline{n\,(r+1-s)}_{t-1} . \frac{u^{t-1}}{\lfloor\underline{t-1}},$$

by the same theorem,

$$= \overset{\infty}{S}_t \frac{u^{t-1}}{\lfloor\underline{t-1}} . S_s^{r+1} \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \lfloor\underline{n\,(r+1-s)}_{t-1} \quad \text{(Art. 16. and 13.)}$$

$$= S_s^{r+1} \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \lfloor\underline{n\,(r+1-s)}_{o}$$

$$+ \overset{\infty}{S}_t \frac{u^t}{\lfloor\underline{t}} . S_s^{r+1} \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \lfloor\underline{n\,(r+1-s)}_{t} \quad \text{(Art. 11.)}$$

$$= (1-1)^r + \overset{\infty}{S}_t \frac{u^t}{\lfloor\underline{t}} \left\{ S^r{}_s \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} . \lfloor\underline{r}_{s-1} . \lfloor\underline{n\,(r+1-s)}_{t} \right.$$

$$\left. + \frac{(-1)^r}{\lfloor\underline{r}} . \lfloor\underline{r} . \lfloor\underline{n.o}_{t} \right\} \quad \text{(Art. 11.)}$$

$$= \overset{\infty}{S}_t \frac{u^t}{\underline{\lfloor t}} \cdot S^r{}_s \frac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,t}$$

$$= S^{r-1}_{t:} \frac{u^t}{\underline{\lfloor t}} \cdot S^r{}_s \frac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,t}$$

$$+ \frac{u^r}{\underline{\lfloor r}} \cdot S^r{}_s \frac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,r}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+r}}{\underline{\lfloor t+r}} \cdot S^r{}_s \frac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,t+r} \quad \text{(Art. 11.).}$$

But $S^r{}_s \dfrac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,t} = 0, \quad \begin{Bmatrix} t=1 \\ t=r-1 \end{Bmatrix};$

and $S^r{}_s \dfrac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,r} = \underline{\lfloor r} \cdot n^r;$

$$\therefore\ (a-1)^r = \overline{un}|^r + \overset{\infty}{S}_t \frac{u^{t+r}}{\underline{\lfloor t+r}} \cdot S^r{}_s \frac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,t+r},$$

$$\text{and } \overline{un}|^r = (a-1)^r - \overset{\infty}{S}_t \frac{u^{t+r}}{\underline{\lfloor t+r}} \cdot S^r{}_s \frac{(-1)^{s-1}}{\underline{\lfloor s-1}} \cdot \underline{\lfloor r}_{\,s-1} \cdot \underline{\lfloor n\,(r+1-s)}_{\,t+r}.$$

Now $a^{\frac{m}{n}} = (1+u)^m = 1 + mu + \overset{\infty}{S}_t \underline{\lfloor m}_{\,t+1} \cdot \dfrac{u^{t+1}}{\underline{\lfloor t+1}},$

and $un = (a-1) - \overset{\infty}{S}_t \dfrac{u^{t+1}}{\underline{\lfloor t+1}} \cdot \underline{\lfloor n}_{\,t+1};$

$$\therefore\ a^{\frac{m}{n}} = 1 + \frac{m}{n} \cdot (a-1) + \overset{\infty}{S}_t \frac{u^{t+1}}{\underline{\lfloor t+1}} \left\{ \underline{\lfloor m}_{\,t+1} - \frac{m}{n} \cdot \underline{\lfloor n}_{\,t+1} \right\}$$

$$= 1 + \frac{m}{n} \cdot (a-1) + \frac{u^2}{\underline{\lfloor 2}} \left\{ \underline{\lfloor m}_{\,2} - \frac{m}{n} \cdot \underline{\lfloor n}_{\,2} \right\}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+2}}{\underline{\lfloor t+2}} \left\{ \underline{\lfloor m}_{\,t+2} - \frac{m}{n} \cdot \underline{\lfloor n}_{\,t+2} \right\}.$$

$$\text{But}\quad \underline{|m}_{\,2} - \frac{m}{n}\cdot\underline{|n}_{\,2} = \underline{\left|\frac{m}{n}\right.}_{\,2}\cdot n^2\,;$$

$$\therefore\quad a^{\frac{m}{n}} = 1 + \frac{m}{n}\cdot(a-1) + \underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{n^2u^2}{\underline{|2}} + \overset{\infty}{S}_t\,\frac{u^{t+2}}{\underline{|t+2}}\left\{\underline{|m}_{\,t+2} - \frac{m}{n}\cdot\underline{|n}_{\,t+2}\right\}.$$

$$\text{Also}\quad n^2u^2 = (a-1)^2 - \overset{\infty}{S}_t\,\frac{u^{t+2}}{\underline{|t+2}}\cdot S^2_s\,\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\underline{|2}_{\,s-1}\cdot\underline{|n\,(3-s)}_{\,t+2}\,;$$

$$\therefore\quad a^{\frac{m}{n}} = 1 + \frac{m}{n}\cdot(a-1) + \underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{(a-1)^2}{\underline{|2}}$$

$$+ \overset{\infty}{S}_t\,\frac{u^{t+2}}{\underline{|t+2}}\left\{\underline{|m}_{\,t+2} - \frac{m}{n}\cdot\underline{|n}_{\,t+2} - \underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{1}{\underline{|2}}\cdot S^2_s\,\frac{(-1)^{s-1}}{s-1}\cdot\underline{|2}_{\,s-1}\cdot\underline{|n\,(3-s)}_{\,t+2}\right\}.$$

Put the coefficient of $\dfrac{u^t}{\underline{|r}} = A_t$,

$$\text{then}\quad A_{t+2} = \underline{|m}_{\,t+2} - \underline{|2\,n}_{\,t+2}\cdot\underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{1}{\underline{|2}} + \underline{|n}_{\,t+2}\left\{\underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{1}{\underline{|2}}\cdot 2 - \frac{m}{n}\right\}$$

$$= \underline{|m}_{\,t+2} - \underline{|2\,n}_{\,t+2}\cdot\underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{1}{\underline{|2}} + \underline{|n}_{\,t+2}\cdot\underline{\left|\frac{m}{n}\right.}\cdot\left(\frac{m}{n} - 2\right)$$

$$= \underline{|m}_{\,t+2} - \underline{\left|\frac{m}{n}\right.}_{\,3}\left(\underline{|2\,n}_{\,t+2}\cdot\frac{1}{\frac{m}{n}-2}\cdot\frac{1}{\underline{|2}} - \underline{|n}_{\,t+2}\cdot\frac{1}{\frac{m}{n}-1}\right)$$

$$= \underline{|m}_{\,t+2} - \underline{\left|\frac{m}{n}\right.}_{\,3}\cdot\frac{1}{\underline{|2}}\cdot S^2_s\,\underline{|n\,(3-s)}_{\,t+2}\cdot\underline{|2}_{\,s-1}\cdot\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\frac{1}{\frac{m}{n}-3-s}\,;$$

$$\therefore\quad a^{\frac{m}{n}} = 1 + \frac{m}{n}\cdot(a-1) + \underline{\left|\frac{m}{n}\right.}_{\,2}\cdot\frac{(a-1)^2}{\underline{|2}}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+2}}{\underline{|t+2}} \left\{ \underline{|m}_{\,t+2} - \underline{\left|\tfrac{m}{n}\right.}_{\,3} \left(\underline{|2n}_{\,t+2} \cdot \frac{1}{\frac{m}{n}-2} \cdot \frac{1}{2} - \underline{|n}_{\,t+2} \cdot \frac{1}{\frac{m}{n}-1} \right) \right\}$$

$$= 1 + \frac{m}{n} \cdot (a-1) + \underline{\left|\tfrac{m}{n}\right.}_{\,2} \cdot \frac{(a-1)^2}{2}$$

$$+ \frac{u^3}{\underline{|3}} \cdot \left\{ \underline{|m}_{\,3} - \underline{\left|\tfrac{m}{n}\right.}_{\,3} \left(\underline{|2n}_{\,3} \cdot \frac{1}{\frac{m}{n}-2} \cdot \frac{1}{2} - \underline{|n}_{\,3} \cdot \frac{1}{\frac{m}{n}-1} \right) \right\}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+3}}{\underline{|t+3}} \left\{ \underline{|m}_{\,t+3} - \underline{\left|\tfrac{m}{n}\right.}_{\,3} \left(\underline{|2n}_{\,t+3} \cdot \frac{1}{\frac{m}{n}-2} \cdot \frac{1}{2} - \underline{|n}_{\,t+3} \cdot \frac{1}{\frac{m}{n}-1} \right) \right\}.$$

But $A_3 = \underline{\left|\tfrac{m}{n}\right.}_{\,3} \cdot n^3$,

and $n^3 u^3 = (a-1)^3 - \overset{\infty}{S}_t \frac{u^{t+3}}{\underline{|t+3}} \cdot S^3_s \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \underline{|3}_{\,s-1} \cdot \underline{|n(4-s)}_{\,t+3}$;

$$\therefore\ a^{\frac{m}{n}} = 1 + \frac{m}{n} \cdot (a-1) + \underline{\left|\tfrac{m}{n}\right.}_{\,2} \cdot \frac{(a-1)^2}{\underline{|2}} + \underline{\left|\tfrac{m}{n}\right.}_{\,3} \cdot \frac{(a-1)^3}{\underline{|3}}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+3}}{\underline{|t+3}} \left\{ \underline{|m}_{\,t+3} - \underline{\left|\tfrac{m}{n}\right.}_{\,3} \cdot \frac{1}{\underline{|2}} \left(\underline{|2n}_{\,t+3} \cdot \frac{1}{\frac{m}{n}-2} - \underline{|n}_{\,t+3} \cdot \frac{2}{\frac{m}{n}-1} \right) \right.$$

$$\left. - \underline{\left|\tfrac{m}{n}\right.}_{\,3} \cdot \frac{1}{\underline{|3}} \cdot S^3_s \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \underline{|3}_{\,s-1} \cdot \underline{|n(4-s)}_{\,t+3} \right\}.$$

Also $A_{t+3} = \underline{|m}_{\,t+3} - \underline{\left|\tfrac{m}{n}\right.}_{\,3} \cdot \frac{1}{\underline{|3}} \left\{ \underline{|3n}_{\,t+3} - \underline{|2n}_{\,t+3} \left(3 - \frac{3}{\frac{m}{2}-2}\right) + \underline{|n}_{\,t+3} \left(3 - \frac{3 \cdot 2}{\frac{m}{n}-1}\right) \right\}$

$$=\underline{|m}_{\,t+3}-\underline{\left|\tfrac{m}{n}\right.}_{\,3}\cdot\frac{1}{\underline{|3}}\left\{\underline{|3n}_{\,t+3}-3\cdot\underline{|2n}_{\,t+3}\cdot\frac{\frac{m}{n}-3}{\frac{m}{n}-2}+3\cdot\underline{|n}_{\,t+3}\cdot\frac{\frac{m}{n}-3}{\frac{m}{n}-1}\right\}$$

$$=\underline{|m}_{\,t+3}-\underline{\left|\tfrac{m}{n}\right.}_{\,4}\cdot\frac{1}{\underline{|3}}\left\{\underline{|3n}_{\,t+3}\cdot\frac{1}{\frac{m}{n}-3}-\underline{|2n}_{\,t+3}\cdot\frac{3}{\frac{m}{n}-2}+\underline{|n}_{\,t+3}\cdot\frac{3}{\frac{m}{n}-1}\right\};$$

$$\therefore\ A_4=\underline{|m}_{\,4}-\underline{\left|\tfrac{m}{n}\right.}_{\,4}\cdot\frac{1}{\underline{|3}}\left\{\underline{|3n}_{\,4}\cdot\frac{1}{\frac{m}{n}-3}-\underline{|2n}_{\,4}\cdot\frac{3}{\frac{m}{n}-2}+\underline{|n}_{\,4}\cdot\frac{3}{\frac{m}{n}-1}\right\}=\underline{\left|\tfrac{m}{n}\right.}_{\,4}\cdot n^4$$

and $a^{\frac{m}{n}}=1+S^3{}_t\,\underline{\left|\tfrac{m}{n}\right.}_{\,t}\cdot\dfrac{(a-1)^t}{\underline{|t}}+\underline{\left|\tfrac{m}{n}\right.}_{\,4}\cdot\dfrac{n^4u^4}{\underline{|4}}$

$$+\overset{\infty}{S}_t\frac{u^{t+4}}{\underline{|t+4}}\left\{\underline{|m}_{\,t+4}-\underline{\left|\tfrac{m}{n}\right.}_{\,4}\cdot\frac{1}{\underline{|3}}\left(\underline{|3n}_{\,t+4}\cdot\frac{1}{\frac{m}{n}-3}-\underline{|2n}_{\,t+4}\cdot\frac{3}{\frac{m}{n}-2}+\underline{|n}_{\,t+4}\cdot\frac{3}{\frac{m}{n}-1}\right)\right\}.$$

But $\overline{nu}|^4=(a-1)^4-\overset{\infty}{S}_t\dfrac{u^{t+4}}{\underline{|t+4}}\cdot S^4{}_s\dfrac{(-1)^{s-1}}{\underline{|s-1}}\cdot\underline{|4}_{\,s-1}\cdot\underline{|n(5-s)}_{\,t+4},$

and $a^{\frac{m}{n}}=1+S^4{}_t\,\underline{\left|\tfrac{m}{n}\right.}_{\,t}\cdot\dfrac{(a-1)^t}{\underline{|t}}$

$$+\overset{\infty}{S}_t\frac{u^{t+4}}{\underline{|t+4}}\left\{\underline{|m}_{\,t+4}-\underline{\left|\tfrac{m}{n}\right.}_{\,4}\cdot\frac{1}{\underline{|4}}\left[\underline{|4n}_{\,t+4}+\underline{|3n}_{\,t+4}\cdot\left(\frac{4}{\frac{m}{n}-3}-4\right)\right.\right.$$

$$\left.\left.-\underline{|2n}_{\,t+4}\cdot\left(\frac{4.3}{\frac{m}{n}-2}-\frac{4.3}{1.2}\right)+\underline{|n}_{\,t+4}\left(\frac{4.3}{\frac{m}{n}-1}-\frac{4.3.2}{1.2.3}\right)\right]\right\}$$

$$=1+S^4{}_t\,\underline{\left|\tfrac{m}{n}\right.}_{\,t}\cdot\frac{(a-1)^t}{\underline{|t}}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+4}}{\underline{|t+4}} \left\{ \underline{|m}_{t+4} - \underline{\left|\frac{m}{n}\right.}_{4} \cdot \frac{1}{\underline{|4}} \left(\underline{|4n}_{t+4} - \underline{|3n}_{t+4} : 4 \cdot \frac{\frac{m}{n} - 4}{\frac{m}{n} - 3} \right.\right.$$

$$\left.\left. + \underline{|2n}_{t+4} \cdot \frac{4 \cdot 3}{1 \cdot 2} \cdot \frac{\frac{m}{n} - 4}{\frac{m}{n} - 2} - \underline{|n}_{t+4} \cdot \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot \frac{\frac{m}{n} - 4}{\frac{m}{n} - 1} \right) \right\}$$

$$= 1 + S^4{}_t \underline{\left|\frac{m}{n}\right.}_t \cdot \frac{(a-1)^t}{\underline{|t}}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+4}}{\underline{|t+4}} \left\{ \underline{|m}_{t+4} - \underline{\left|\frac{m}{n}\right.}_{5} \cdot \frac{1}{\underline{|4}} \left(\underline{|4n}_{t+4} \cdot \frac{1}{\frac{m}{n} - 4} - \underline{|3n}_{t+4} \cdot \frac{4}{\frac{m}{n} - 3} \right.\right.$$

$$\left.\left. + \underline{|2n}_{t+4} \cdot \frac{\frac{4 \cdot 3}{1 \cdot 2}}{\frac{m}{n} - 2} - \underline{|n}_{t+4} \cdot \frac{\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}}{\frac{m}{n} - 1} \right) \right\}$$

$$= 1 + S^4{}_t \underline{\left|\frac{m}{n}\right.} \cdot \frac{(a-1)^t}{\underline{|t}}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+4}}{\underline{|t+4}} \left\{ \underline{|m}_{t+4} - \underline{\left|\frac{m}{n}\right.}_{5} \cdot \frac{1}{\underline{|4}} \cdot S^4{}_s \underline{|n\,(5-s)}_{t+4} \cdot \underline{|4}_{s-1} \cdot \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \frac{1}{\frac{m}{n} - \overline{5-s}} \right. .$$

Hence, *probably,*

$$a^{\frac{m}{n}} = 1 + S^r{}_t \underline{\left|\frac{m}{n}\right.}_t \cdot \frac{(a-1)^t}{\underline{|t}}$$

$$+ \overset{\infty}{S}_t \frac{u^{t+r}}{\underline{|t+r}} \left\{ \underline{|m}_{t+r} - \underline{\left|\frac{m}{n}\right.}_{r+1} \cdot \frac{1}{\underline{|r}} \cdot S^r{}_s \underline{|n\,(r+1-s)}_{t+r} \cdot \underline{|r}_{s-1} \cdot \frac{(-1)^{s-1}}{\underline{|s-1}} \cdot \frac{1}{\frac{m}{n} - \overline{r+1-s}} \right\} .$$

Suppose this true for r eliminations,

$$\text{then}\quad a^{\frac{m}{n}} = 1 + S^r\, \underline{|\tfrac{m}{n}}_{\,t} \cdot \frac{(a-1)^t}{\underline{|t}}$$

$$+\frac{u^{r+1}}{\underline{|r+1}}\left\{\underline{|m}_{\,r+1} - \underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{1}{\underline{|r}}\cdot S^r{}_s\, \underline{|n(r+1-s)}_{\,r+1}\cdot \underline{|r}_{\,s-1}\cdot\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\frac{1}{\frac{m}{n}-\overline{r+1-s}}\right\}$$

$$+\overset{\infty}{S_t}\frac{u^{t+r+1}}{\underline{|t+r+1}}\left\{\underline{|m}_{\,t+r+1} - \underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{1}{\underline{|r}}\cdot S^r{}_s\, \underline{|n(r+1-s)}_{\,t+r+1}\cdot \underline{|r}_{\,s-1}\cdot\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\frac{1}{\frac{m}{n}-\overline{r+1-s}}\right\}$$

$$= 1 + S^r{}_t\, \underline{|\tfrac{m}{n}}_{\,t}\cdot\frac{(a-1)^t}{\underline{|t}} + \underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{\overline{un}|^{r+1}}{\underline{|r+1}}$$

$$+\overset{\infty}{S_t}\frac{u^{t+r+1}}{\underline{|t+r+1}}\left\{\underline{|m}_{\,t+r+1} - \underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{1}{\underline{|r}}\cdot S^r{}_s\, \underline{|n(r+1-s)}_{\,t+r+1}\cdot \underline{|r}_{\,s-1}\cdot\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\frac{1}{\frac{m}{n}-\overline{r+1-s}}\right\},$$

$$\text{and}\quad \overline{un}|^{r+1} = (a-1)^{r+1} - \overset{\infty}{S_t}\frac{u^{t+r+1}}{\underline{|t+r+1}}\cdot S_s^{r+1}\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\underline{|r+1}_{\,s-1}\cdot\underline{|n(r+2-s)}_{\,t+r+1}$$

$$\therefore\quad \underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{\overline{vn}|^{r+1}}{\underline{|r+1}} = \underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{(a-1)^{r+1}}{\underline{|r+1}}$$

$$-\underline{|\tfrac{m}{n}}_{\,r+1}\cdot\frac{1}{\underline{|r+1}}\cdot\overset{\infty}{S_t}\frac{u^{t+r+1}}{\underline{|t+r+1}}\cdot S_s^{r+1}\frac{(-1)^{s-1}}{\underline{|s-1}}\cdot\underline{|r+1}_{\,s-1}\cdot\underline{|n(r+2-s)}_{\,t+r+1}.$$

$$\text{Whence}\quad a^{\frac{m}{n}} = 1 + S_t^{r+1}\,\underline{|\tfrac{m}{n}}_{\,t}\cdot\frac{(a-1)^t}{\underline{|t}} + \overset{\infty}{S_t}\frac{u^{t+r+1}}{\underline{|t+r+1}}\cdot A_{t+r+1}.$$

Where $A_{t+r+1} = |\underline{m}_{\,t+r+1} - \dfrac{|\underline{\tfrac{m}{n}}_{\,r+1} \cdot |\underline{n\,(r+1)}_{\,t+r+1}}{|\underline{r+1}}$

$$- |\underline{\tfrac{m}{n}}_{\,r+1} \cdot \frac{1}{|\underline{r}} \cdot S^{r}_{s}\, |\underline{n\,(r+1-s)}_{\,t+r+1} \cdot |\underline{r}_{\,s-1} \cdot \frac{(-1)^{s-1}}{|\underline{s-1}} \left\{ \frac{1}{\frac{m}{n} - \overline{r+1-s}} - \frac{1}{r+1} \cdot \frac{1}{s} \cdot (r+1) \right\}$$

$$= |\underline{m}_{\,t+r+1} - \frac{|\underline{\tfrac{m}{n}}_{\,r+1} \cdot |\underline{n\,(r+1)}_{\,t+r+1}}{|\underline{r+1}}$$

$$- |\underline{\tfrac{m}{n}}_{\,r+1} \cdot \frac{1}{|\underline{r}} \cdot S^{r}_{s}\, |\underline{n\,(r+1-s)}_{\,t+r+1} \cdot |\underline{r}_{\,s-1} \cdot \frac{(-1)^{s}}{|\underline{s}} \cdot \frac{\frac{m}{n} - \overline{r+1-s} - s}{\frac{m}{n} - \overline{r+1-s}}$$

$$= |\underline{m}_{\,t+r+1} - \frac{|\underline{\tfrac{m}{n}}_{\,r+1} \cdot |\underline{n\,(r+1)}_{\,t+r+1}}{|\underline{r+1}}$$

$$- |\underline{\tfrac{m}{n}}_{\,r+2} \cdot \frac{1}{|\underline{r+1}}\, S^{r}_{s}\, |\underline{n\,(r+1-s)}_{\,t+r+1} \cdot |\underline{r+1}_{\,s} \cdot \frac{(-1)^{s}}{|\underline{s}} \cdot \frac{1}{\frac{m}{n} - \overline{r+1-s}}$$

$$= |\underline{m}_{\,t+r+1} - |\underline{\tfrac{m}{n}}_{\,r+2} \cdot \frac{1}{|\underline{r+1}} \cdot S_{s}^{r+1}\, |\underline{n\,(r+2-s)}_{\,t+r+1} \cdot |\underline{r+1}_{\,s-1} \cdot \frac{(-1)^{s-1}}{|\underline{s-1}} \cdot \frac{1}{\frac{m}{n} - \overline{r+2-s}},$$

and $a^{\frac{m}{n}} = 1 + S_{t}^{r+1} |\underline{\tfrac{m}{n}}_{\,t} \cdot \dfrac{(a-1)^{t}}{|\underline{t}} + \overset{\infty}{S_{t}} \dfrac{u^{t+r+1}}{|\underline{t+r+1}}$

$$\left\{ |\underline{m}_{\,t+r+1} - |\underline{\tfrac{m}{n}}_{\,r+2} \cdot \frac{1}{|\underline{r+1}} \cdot S_{s}^{r+1}\, |\underline{n\,(r+2-s)}_{\,t+r+1} \cdot |\underline{r+1}_{\,s-1} \cdot \frac{(-1)^{s-1}}{|\underline{s-1}} \cdot \frac{1}{\frac{m}{n} - \overline{r+2-s}} \right\}.$$

Hence, if the supposed law holds for r eliminations, it will hold for $r+1$: but it does hold for 4; therefore it will hold for 5, 6, &c. and hence it is *generally* true.

And thus finally:

$$a^{\frac{m}{n}} = 1 + S^r{}_t \underline{\left\lfloor \frac{m}{n} \right.}_{t} \cdot \frac{(a-1)^t}{\lfloor\underline{t}} + \overset{\infty}{S}_t \frac{(a^{\frac{1}{n}}-1)^{t+r}}{\lfloor\underline{t+r}}$$

$$\left\{ \lfloor\underline{m}_{t+r} - \underline{\left\lfloor \frac{m}{n} \right.}_{r+1} \cdot \frac{1}{\lfloor\underline{r}} \cdot S^r{}_s \lfloor\underline{n\,(r+1-s)}_{t+r} \cdot \lfloor\underline{r}_{s-1} \cdot \frac{(-1)^{s-1}}{\lfloor\underline{s-1}} \cdot \frac{1}{\frac{m}{n} - \overline{r+1-s}} \right\} .$$

Ex. 5. **Taylor's Theorem:** $E_x \,.\, u = \overset{\infty}{S}_m \overline{Dx}\rvert^{m-1} d_x{}^{m-1} : u.$

For, whatever the form of u may be, it can be reduced into a series of monomials, into none of which x enters as an exponent. We may therefore assume

$$u = S^r_{\iota n} a_n x^{a_n} \ldots\ldots\ldots\ldots (1);$$

$$\therefore\ E_x \,.\, u = S^r{}_n a_n \,.\, \overline{x + Dx}\rvert^{a_n}$$

$$= S^r{}_n a_n \,.\, \overset{\infty}{S}_m \lfloor\underline{a_n}_{m-1} \cdot \frac{1}{\lfloor\underline{m-1}} \,.\, x^{a_n - m + 1} \overline{Dx}\rvert^{m-1},$$

by the binomial theorem

$$= S^r{}_n \overset{\infty}{S}_m a_n \cdot \lfloor\underline{a_n}_{m-1} \cdot \frac{1}{\lfloor\underline{m-1}} \,.\, x^{a_n - m + 1} \,.\, \overline{Dx}\rvert^{m-1} \text{ (Art. 13.)}$$

$$= \overset{\infty}{S}_m S^r{}_n a_n \,.\, \lfloor\underline{a_n}_{m-1} \cdot \frac{1}{\lfloor\underline{m-1}} \,.\, x^{a_n - m + 1} \,.\, \overline{Dx}\rvert^{m-1} \text{ (Art. 15.)}$$

$$= \overset{\infty}{S}_m \frac{\overline{Dx}\rvert^{m-1}}{\lfloor\underline{m-1}} \,.\, S^r{}_n a_n \cdot \lfloor\underline{a_n}_{m-1}\, x^{a_n - m + 1} \text{ (Art. 13.)}$$

$$= \overset{\infty}{S}_m \frac{\overline{Dx}|^{m-1}}{\lfloor m-1} \,.\, d_x^{m-1} \,.\, S^r{}_n a_n x^n.$$

$$= \overset{\infty}{S}_m \frac{\overline{Dx}|^{m-1}}{\lfloor m-1} \,.\, d_x^{m-1} u, \text{ by (1),}$$

$$= \overset{\infty}{S}_m \overline{Dx}|^{m-1} d_x^{m-1} : u.$$

Cor. For u write $E_y \,.\, u$,

$$\text{then } E_x \,.\, E_y \,.\, u = \overset{\infty}{S}_m \overline{Dx}|^{m-1} d_x^{m-1} : E_y \,.\, u$$

$$= \overset{\infty}{S}_m \overline{Dx}|^{m-1} d_x^{m-1} : \overset{\infty}{S}_m \overline{Dy}|^{n-1} d_y^{n-1} : u$$

$$= \overset{\infty}{S}_m \overline{Dx}|^{m-1} \overset{\infty}{S}_n \overline{Dy}|^{n-1} d_x^{m-1} : d_y^{n-1} : u$$

$$= \overset{\infty}{S}_m S^m{}_n \overline{Dx}|^{m-n} \overline{Dy}|^{n-1} d_x^{m-n} : d_y^{n-1} : u, \text{ (Art. 18.);}$$

and in the same manner we shall find

$$E_x \,.\, E_y \,.\, E_z \,.\, u = \overset{\infty}{S}_m S^m{}_n S^n{}_r \overline{Dx}|^{m-n} \,.\, \overline{Dy}|^{n-r} \,.\, \overline{Dz}|^{r-1} d_x^{m-n} : d_y^{n-r} : d_z^{r-1} : u,$$

and so for any number of variables.

Ex. 6. Maclaurin's Theorem: $u = \overset{\infty}{S}_m x^{m-1} d_{x,0}{}^{m-1} : u.$

Assume $u = \overset{\infty}{S}_n a_{n-1} x^{n-1}$ (1), where a_{n-1} is either zero or some finite quantity.

Then $d^m{}_x u = \overset{\infty}{S}_n a_{n-1} \,.\, \underset{m}{\underline{\lfloor n-1}} \,.\, x^{n-m-1}$

$$= S^m{}_n a_{n-1} \,.\, \underset{m}{\underline{\lfloor n-1}} \,.\, x^{n-m-1} + a_m \,.\, \underset{m}{\underline{\lfloor m}} \,.\, x^0 + \overset{\infty}{S}_n a_{m+n} \,.\, \underset{m}{\underline{\lfloor m+n}} \,.\, x^n.$$

But $\underline{|n-1}_{m} = 0$, from $n=1$ to $n=m$;

$$\therefore\ d^m{}_x u = a_m \,.\, \underline{|m} + \overset{\infty}{S}_n\, a_{m+n} \,.\, \underline{|m+n}_{m}\, x^n,$$

$$\text{and } d^m{}_{x,0} u = a_m \,.\, \underline{|m}, \quad \text{or } a_m = d^m{}_{x,0} : u.$$

$$\text{Also } a_0 = d^0{}_{x,0} : u; \quad \therefore\ a_{m-1} = d_{x,0}{}^{m-1} : u, \quad \left\{\begin{matrix} m=1 \\ m=\infty \end{matrix}\right\}.$$

Hence, substituting in (1), $\quad u = \overset{\infty}{S}_m\, x^{m-1} d_{x,0}{}^{m-1} : u.$

Ex. 7. Maclaurin's Theorem applied to a function of two variables:

$$u = \overset{\infty}{S}_m\, x^{m-1} \,.\, \overset{\infty}{S}_n\, y^{n-1} \,.\, d_{x,0}{}^{m-1} : d_{y,0}{}^{n-1} : u.$$

Assume $\quad u = \overset{\infty}{S}_m\, x^{m-1} \,.\, \overset{\infty}{S}_n\, a_{m-1,\, n-1}\, y^{n-1},$

where $a_{m-1,\, n-1}$ is not infinite.

$$\text{Then } d^s{}_y u = \overset{\infty}{S}_m\, x^{m-1} \,.\, \overset{\infty}{S}_n\, a_{m-1,\, n-1} \,.\, \underline{|n-1}_{s} \,.\, y^{n-s-1};$$

$$\therefore\ d^s{}_{y,0} u = \overset{\infty}{S}_m\, x^{m-1} \,.\, a_{m-1,\, s} \,.\, \underline{|s}, \text{ as in the last example,}$$

$$\text{or } d^n{}_{y,0} : u = \overset{\infty}{S}_m\, x^{m-1}\, a_{m-1,\, n}.$$

$$\text{Also } d^0{}_{y,0} : u = \overset{\infty}{S}_m\, x^{m-1}\, a_{m-1,\, 0};$$

$$\therefore\ d_{y,0}{}^{n-1} : u = \overset{\infty}{S}_m\, x^{m-1}\, a_{m-1,\, n-1}, \quad \left\{\begin{matrix} m=1 \\ n=\infty \end{matrix}\right\}.$$

$$\text{Hence } d^r{}_x \,.\, d_{y,0}{}^{n-1} : u = \overset{\infty}{S}_m\, \underline{|m-1}_{r} \,.\, x^{m-r-1} \,.\, a_{m-1,\, n-1},$$

$$\text{and } d^r{}_{x,0} \,.\, d_{y,0}{}^{n-1} : u = \overset{\infty}{S}_m\, \underline{|r} \,.\, a_{r,\, n-1}, \text{ as in the last example.}$$

$$\text{Also } d^0{}_{x,0} \,.\, d_{y,0}{}^{n-1} : u = a_{0,\, n-1},$$

$$\therefore\ d_{x,0}{}^{m-1} : d_{y,0}{}^{n-1} : u = a_{m-1,\, n-1} \quad \left\{\begin{matrix} m=1 \\ m=\infty \end{matrix}\right\} \text{ and } \left\{\begin{matrix} n=1 \\ n=\infty \end{matrix}\right\}.$$

Whence, substituting in (1),

$$u = \overset{\infty}{S}_m x^{m-1} \overset{\infty}{S}_n y^{n-1} d_{x,0}{}^{m-1} : d_{y,0}{}^{n-1} : u.$$

Ex. 8. To expand $\overline{1+e.\cos x}|^n$ in a series ascending by cosines of the multiples of x.

$$\overline{1+e\cos x}|^n = 1 + \overset{\infty}{S}_m \lfloor\underline{n}_{m} . \frac{\overline{e\cos x}|^m}{\lfloor\underline{m}}, \text{ by the binomial theorem,}$$

$$= 1 + \overset{\infty}{S}_m \left\{ \lfloor\underline{n}_{2m-1} . \frac{\overline{e\cos x}|^{2m-1}}{\lfloor\underline{2m-1}} + \lfloor\underline{n}_{2m} . \frac{\overline{e\cos x}|^{2m}}{\lfloor\underline{2m}} \right\} \quad \text{(Art. 12.)}$$

$$= 1 + \overset{\infty}{S}_m \left\{ \lfloor\underline{n}_{2m-1} . \frac{e^{2m-1}}{\lfloor\underline{2m-1}} . \frac{1}{2^{2m-2}} . S^m{}_r \lfloor\underline{2m-1}_{m-r} . \frac{1}{\lfloor\underline{m-r}} . \cos \overline{2r-1} . x \right.$$

$$\left. + \lfloor\underline{n}_{2m} . \frac{e^{2m}}{\lfloor\underline{2m}} . \left(\frac{1}{2^{2m-1}} . S^m{}_r \lfloor\underline{2m}_{m-r} . \frac{1}{\lfloor\underline{m-r}} . \cos 2rx + \lfloor\underline{2m}_{m} . \frac{1}{2^{2m} . \lfloor\underline{m}} \right) \right\},$$

by substitution,

$$= 1 + \overset{\infty}{S}_m \lfloor\underline{n}_{2m} . \lfloor\underline{2m}_{m} . \frac{e^{2m}}{2^{2m} . \lfloor\underline{2m} . \lfloor\underline{m}}$$

$$+ \overset{\infty}{S}_m \left\{ \lfloor\underline{n}_{2m-1} . \frac{e^{2m-1}}{2^{2m-2} . \lfloor\underline{2m-1}} . S^m{}_r \lfloor\underline{2m-1}_{m-r} . \frac{\cos \overline{2r-1} . x}{\lfloor\underline{m-r}} \right.$$

$$\left. + \lfloor\underline{n}_{2m} . \frac{e^{2m}}{2^{2m-1} . \lfloor\underline{2m}} . S^m{}_r \lfloor\underline{2m}_{m-r} . \frac{\cos 2rx}{\lfloor\underline{m-r}} \right\}.$$

But $\lfloor\underline{2m}_{m} . \dfrac{1}{\lfloor\underline{2m} . \lfloor\underline{m}} = \dfrac{1}{(\lfloor\underline{m})^2}$; whence, and by Art. 16,

$$\overline{1+e\cos x}|^n = 1 + \overset{\infty}{S}_m \left\{ \lfloor\underline{n}_{2m} . \frac{e^{2m}}{\lfloor\underline{2^{2m}} . (\lfloor\underline{m})^2} \right.$$

$$+ \cos \overline{2m-1} . x . \overset{\infty}{S}_r \lfloor\underline{n}_{2m+2r-3} . \lfloor\underline{2m+2r-3}_{r-1} . \frac{e^{2m+2r-3}}{2^{2m+2r-4} . \lfloor\underline{2m+2r-3} . \lfloor\underline{r-1}}$$

$$+ \cos 2mx \,.\, \overset{\infty}{S}_r \underline{|n}_{2m+2r-2} \,.\, \underline{|2m+2r-2}_{r-1} \,.\, \frac{e^{2m+2r-2}}{2^{2m+2r-3} \,.\, \underline{|2m+2r-2} \,.\, \underline{|r-1}} \Bigg\}$$

$$= 1 + \overset{\infty}{S}_m \left\{ \underline{|n}_{2m} \,.\, \frac{e^{2m}}{2^{2m} \,.\, (\underline{|m})^2} \right.$$

$$\left. + \cos mx \,.\, \overset{\infty}{S}_r \underline{|n}_{m+2r-2} \,.\, \underline{|m+2r-2}_{r-1} \,.\, \frac{e^{m+2r-2}}{2^{m+2r-3} \,.\, \underline{|m+2r-2}} \,.\, \frac{1}{\underline{|r-1}} \right\},$$

by the converse of Art. 12.

Ex. 9. Laplace's Theorem: If

$$y = \psi\,\{z + x \,.\, \phi(y)\},$$

when z is independent of x and y, then shall

$$f(y) = f\psi(z) + \overset{\infty}{S}_m \frac{x^m}{\underline{|m}} \,.\, d_z^{\,m-1} \,.\, \{\overline{\phi\psi(z)}|^m \,.\, d_z \,.\, f\psi(z)\}.$$

For, by Maclaurin's theorem,

$$f(y) = d^0_{x,0} \,.\, f(y) + \overset{\infty}{S}_m \frac{x^m}{\underline{|m}} \,.\, d^m_{x,0} \,.\, f(y).$$

Put $z + x \,.\, \phi(y) = u$, then $y = \psi(u)$,

$$\text{and } \frac{d_x y}{d_z y} = \frac{d_x \,.\, \psi(u)}{d_z \,.\, \psi(u)} = \frac{d_x u \,.\, d_u \,.\, \psi(u)}{d_z u \,.\, d_u \,.\, \psi(u)} = \frac{d_x u}{d_z u} = \frac{d_x\,(z + x \,.\, \phi(y))}{d_z\,(z + x \,.\, \phi(y))}$$

$$= \frac{\phi(y) + x \,.\, d_x \,.\, \phi(y)}{1 + x \,.\, d_z \,.\, \phi(y)} = \frac{\phi(y) + x \,.\, d_x y \,.\, d_y \,.\, \phi(y)}{1 + x \,.\, d_z y \,.\, d_y \,.\, \phi(y)}.$$

Whence, mnltiplying up,

$$d_x y\,\{1 + x d_z y \,.\, d_y \,.\, \phi(y)\} = d_z y\,\{\phi(y) + x \,.\, d_x y \,.\, d_y \,.\, \phi(y)\};$$

and, cancelling identical terms, $d_x y = d_z y \,.\, \phi(y)$.

$$\text{Again, } d_x \,.\, f(y) = d_x y \,.\, d_y f(y)$$

$$= d_z y \,.\, \phi(y) \,.\, d_y f(y),$$

$$d_x{}^2 \cdot f(y) = d_x \{d_z y \cdot \phi(y) \cdot d_y \cdot f(y)\}$$

$$= d_z \{d_x y \cdot \phi(y) \cdot d_y \cdot f(y)\}$$

$$= d_z \cdot \{d_z y \cdot \overline{\phi(y)}|^2 \cdot d_y \cdot f(y)\},$$

$$d^3{}_x \cdot f(y) = d_x \cdot d_z \{d_z y \cdot \overline{\phi(y)}|^2 \cdot d_y \cdot f(y)$$

$$= d_z \cdot d_x \{d_z y \cdot \overline{\phi(y)}|^2 \cdot d_y \cdot f(y)\}$$

$$= d^2{}_z \cdot \{d_x y \cdot \overline{\phi(y)}|^2 \cdot d_y \cdot f(y)\}$$

$$= d^2{}_z \cdot \{d_z y \cdot \overline{\phi(y)}|^3 \cdot d_y \cdot f(y)\},$$

and similarly, $d^m{}_x \cdot f(y) = d_z{}^{m-1} \cdot \{d_z y \cdot \overline{\phi(y)}|^m \cdot d_y \cdot f(y)\}$

$$= d_z{}^{m-1} \{\phi(y)^m \cdot d_z \cdot f(y)\}.$$

Put $x = 0$, and y becomes $\psi(z)$;

$$\therefore\ d^m{}_{x,0} \cdot f(y) = d_z{}^{m-1} \cdot \{\overline{\phi\psi(z)}|^m \cdot d_z \cdot f\psi(z)\},$$

$$\text{and } f(y) = f\psi(z) + \overset{\infty}{S}_m \frac{x^m}{\underline{|m}} \cdot d_z{}^{m-1} \cdot \{\overline{\phi\psi(z)}|^m \cdot d_z \cdot f\psi(z)\}.$$

COR. Put $\psi(u) = u$, and we shall get

$$f(y) = f(z) + \overset{\infty}{S}_m \frac{x^m}{\underline{|m}} \cdot d_z{}^{m-1} \cdot \{\overline{\phi(z)}|^m \cdot d_z \cdot f(z)\},$$

if $y = z + x \cdot \phi(y)$;

which is Lagrange's theorem.

Ex. 10. If x, x', z, and z' are independent of each other, while y and y' are determined by the equations

$$y = \psi\{z + x \cdot \phi(y)\},$$

$$\text{and } y' = \psi'\{z' + x' \cdot \phi'(y)\};$$

then shall $f(y, y') = f(\psi z, \psi' z')$

$$+ \overset{\infty}{S}_m \left[\frac{x^m}{\underline{|m}} . d_z^{m-1} . \{ \overline{\phi \psi z}|^m . d_z . f(\psi z, \psi' z') \} \right.$$

$$\left. + \frac{x'^m}{\underline{|m}} . d_{z'}^{m-1} \{ \overline{\phi' \psi' z'}|^m \, d_{z'} . f(\psi z, \psi', z') \} \right]$$

$$+ \overset{\infty}{S}_m \overset{\infty}{S}_n \frac{x^m \, x'^n}{\underline{|m} . \underline{|n}} . d_z^{m-1} . d_{z'}^{n-1} \{ \overline{\phi \psi z}|^m . \overline{\phi' \psi' z'}|^{n'} . d_z . d_{z'} f(\psi z, \psi' z') \} \quad (1).$$

For, $f(y, y') = \overset{\infty}{S}_m \overset{\infty}{S}_n \dfrac{x^{m-1} \, x'^{n-1}}{\underline{|m-1} . \underline{|n-1}} . d_{x,0}^{m-1} . d_{x',0}^{n-1} . f(y, y')$ (Ex. 7.)

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\underline{|m-1}} . d_{x,0}^{m-1} . d^0_{x',0} . f(y, y')$$

$$+ \overset{\infty}{S}_m \overset{\infty}{S}_n \frac{x^{m-1} \, x'^n}{\underline{|m-1} . \underline{|n}} . d_{x,0}^{m-1} . d^n_{x',0} . f(y, y') \quad \text{(Art. 11.)}$$

$$= d^0_{x,0} . d^0_{x',0} . f(y, y')$$

$$+ \overset{\infty}{S}_m \frac{x^m}{\underline{|m}} . d^m_{x,0} . d^0_{x',0} . f(y, y') + \overset{\infty}{S}_n \frac{x'^n}{\underline{|n}} . d^0_{x,0} . d^n_{x',0} . f(y, y')$$

$$+ \overset{\infty}{S}_m \overset{\infty}{S}_n \frac{x^m . x'^n}{\underline{|m} . \underline{|n}} . d^m_{x,0} . d^n_{x',0} . f(y, y'), \quad \text{(Art. 11.)}$$

But, as in the last example, we shall find

$$d^m_x . f(y, y') = d_z^{m-1} . \{ \overline{\phi y}|^m . d_z . f(y, y') \} ;$$

$$\therefore \; d^m_x . d^n_{x'} . f(y, y') = d_z^{m-1} \{ \overline{\phi y}|^m \, d_z . d^n_{x'} . f(y, y') \}$$

$$= d_z^{m-1} \{ \overline{\phi y}|^m . d_z . d_{z'}^{n-1} . [\overline{\phi' y'}|^n \, d_{z'} . f(y, y')] \}$$

$$= d_z^{m-1} . d_{z'}^{n-1} \{ \overline{\phi y}|^m . \overline{\phi' y'}|^n . d_z . d_{z'} f(y, y') \} ;$$

since y and z are independent of z', and y' of z.

$$\therefore\ d_{z,0}^{\ m} \,.\, d_{z',0}^{\ n} \,.\, f(y, y')$$

$$= d_z^{\ m-1} \,.\, d_{z'}^{\ n-1} \,.\, \{\overline{\phi\psi z}|^m \,.\, \overline{\phi\psi' z'}|^n\, d_z \,.\, d_{z'} \,.\, f(\psi z, \psi' z')\}\,;$$

whence, by substitution, we get the equation (1).

Ex. 11. Given $u = nt + e\,.\sin u$, to develope u in powers of e.

By Lagrange's Theorem, if $y = z + x\,.\,\phi(y)$,

$$\text{then } f(y) = f(z) + \overset{\infty}{S_m} \frac{x^m}{\lfloor m} \,.\, d_z^{\ m-1} \,.\, \{\overline{\phi z}|^m \,.\, d_z \,.\, f(z)\}.$$

Put, therefore, $y = u$, $z = nt$, $x = e$, $\phi(y) = \sin y$, and $f(y) = f(u) = u$.

Then $f(z) = z$, and $d_z\,.\,f(z) = 1$.

$$\text{Whence } u = nt + \overset{\infty}{S_m} \frac{e^m}{\lfloor m} \,.\, d_{z,\,nt}^{\ m-1} \,.\, \overline{\sin z}|^m$$

$$= nt + \overset{\infty}{S_m} \frac{e^{2m-1}}{\lfloor 2m-1} \,.\, d_{z,\,nt}^{\ 2m-2} \,.\, \overline{\sin z}|^{2m-1} + \overset{\infty}{S_m} \frac{e^{2m}}{\lfloor 2m} \,.\, d_{z,\,nt}^{\ 2m-1} \,.\, \overline{\sin z}|^{2m},$$

(Art. 12.)

$$\text{But } \overline{\sin z}|^{2m-1} = \frac{1}{(-4)^{m-1}} \,.\, S^m{}_r \underset{m-r}{\lfloor 2m-1} \,.\, \frac{(-1)^{m-r}}{\lfloor m-r} \,.\sin \overline{2r-1}\,.\,z,$$

$$\text{and } \overline{\sin z}|^{2m} = \frac{2}{(-4)^m} \,.\, S^m{}_r \underset{m-r}{\lfloor 2m} \,.\, \frac{(-1)^{m-r}}{\lfloor m-r} \,.\cos 2rz + \underset{m}{\lfloor 2m} \,.\, \frac{1}{4^m \,.\lfloor m}\,.$$

Also $d_z^{\ 2m-2}\,.\sin az = (-1)^{m-1}\,.\,a^{2m-2}\,.\sin az,$

and $d_z^{\ 2m-1}\,.\cos az = (-1)^m\,.\,a^{2m-1}\,.\sin az\,;$

$$\therefore\ d_z^{\ 2m-2} \,.\, \overline{\sin z}|^{2m-1}$$

$$= \frac{1}{(-4)^{m-1}} \,.\, S_r{}^m \underset{m-r}{\lfloor 2m-1} \,.\, \frac{(-1)^{m-r}}{\lfloor m-r} \,.\, \overline{2r-1}|^{2m-2} \,.\, (-1)^{m-1} \,.\sin \overline{2r-1}\,.\,z,$$

$$= \frac{1}{(-4)^{m-1}} \,.\, S^m{}_r \underset{m-r}{\lfloor 2m-1} \,.\, \frac{(-1)^{r-1}}{\lfloor m-r} \,.\, \overline{2r-1}|^{2m-2} \,.\sin \overline{2r-1}\,.\,z,$$

and $d_z^{2m-1} \,.\, \overline{\sin z}|^{2m}$

$$= \frac{2}{(-4)^m} \,.\, S^m{}_r \underline{|2m}_{\,m-r} \,.\, \frac{(-1)^{m-r}}{\underline{|m-r}} \,.\, (-1)^m \,.\, \overline{2r}|^{2m-1} \,.\, \sin 2rz$$

$$= \frac{2}{(-4)^m} \,.\, S^m{}_r \underline{|2m}_{\,m-r} \,.\, \frac{(-1)^{r}}{\underline{|m-r}} \,.\, \overline{2r}|^{2m-1} \,.\, \sin 2rz.$$

Whence, substituting these values, and putting $z = nt$,

$$u = nt + \overset{\infty}{S}_m \frac{e^{2m-1}}{(-4)^{m-1} \,.\, \underline{|2m-1}} \,.\, S^m{}_r \underline{|2m-1}_{\,m-r} \,.\, \frac{(-1)^{r-1}}{\underline{|m-r}} \,.\, \overline{2r-1}|^{2m-2} \,.\, \sin \overline{2r-1} \,.\, nt$$

$$+ \overset{\infty}{S}_m \frac{e^{2m}}{(-4)^{m} \,.\, \underline{|2m}} \,.\, S^m{}_r \,.\, \underline{|2m}_{\,m-r} \,.\, \frac{(-1)^{r}}{\underline{|m-r}} \,.\, \overline{2r}|^{2m-1} \,.\, \sin 2rnt.$$

34. These examples are probably sufficient to show the advantages that may be derived from the proposed notation. I shall therefore conclude with a comparison between it and that at present used.

1. $P^n{}_m \,.\, a_m = a_1 \,.\, a_2 \,.\, a_3 \ldots a_n$;

$\underline{|n}_{\,m,\pm r} = n \,.\, (n \pm r)\,(n \pm 2r) \ldots (n \pm \overline{m-1} \,.\, r)$;

$\underline{|n}_{\,m} = n\,(n-1) \ldots (n-m+1)$, and $\underline{|n} = 1 \,.\, 2 \,.\, 3 \ldots n$.

2. $S^n{}_m a_m = a_1 + a_2 + a_3 + \&c. + a_n$.

3. $S^r{}_m S^s{}_n a_{m,n} = a_{1,1} + a_{1,2} + a_{1,3} + \&c. + a_{1,s}$

$+ a_{2,1} + a_{2,2} + a_{2,3} + \&c. + a_{2,s}$

$+ a_{3,1} + a_{3,2} + a_{3,3} + \&c. + a_{3,s}$

$+ \&c. \qquad + \&c.$

$+ a_{r,1} + a_{r,2} + a_{r,3} + \&c. + a_{r,s}$.

4. $C_r^{m,n} \,.\, a_r$ = sum of every combination of $a_1, a_2, a_3, \ldots a_n$, taken m at a time.

5. $D_x . u = \Delta u$, x only being made to vary.

6. $\Delta_x . u = \Delta u$, the increment of x being unity.

7. If $u = f(x)$, then $E_x . u = f(x + Dx)$.

8. $d^m{}_x u = \dfrac{d^m u}{d x^m}$, $d^m{}_x . d^n{}_y u = \dfrac{d^{m+n} . u}{dx^m . dy^n}$.

9. $d^m{}_x : u = \dfrac{d^m u}{d x^m} . \dfrac{1}{1 . 2 . 3 \ldots m}$,

$$d^m{}_x \because d^n{}_y : u = \frac{d^{m+n} u}{dx^m . dy^n} . \frac{1}{1 . 2 . 3 \ldots m . 1 . 2 . 3 \ldots n}.$$

10. $d^n{}_{x,a} u = \dfrac{d^n u}{dx^n}$, x being put $= a$ after the differentiations have been all performed.

11. $(\int_{x,b} - \int_{x,a})\, u = \int u dx$, between the limits $x = a$ and $x = b$.

T. JARRETT.

CATH. HALL,
Nov. 10, 1827.

NOTES.

Art. 2. Through a mistake of the Compositor, the indices belonging to the symbols of operation are not placed in the manner originally intended; instead of $A^n{}_m$ the symbol should have been $\overset{n}{\underset{m}{A}}$.

Art. 8. The following Theorems are easily demonstrated, and are of constant application:

1. $\underline{|n}_{m,r} = \underline{|n}_{s,r} \cdot \underline{|n + rs}_{m-s,r}.$

2. $\underline{|n}_{m,r} = \underline{|n + \overline{m-1}\,.\,r}_{m,-r}.$

3. $\underline{|n}_{m,r} = \underline{|ns}_{m,rs} \cdot \dfrac{1}{s^m}.$

4. $\underline{|n}_{o,r} = 1$, and $\underline{|o} = 1.$

Art. 12. If the series is not infinite, it is obvious that we shall have

$$\overset{2n}{\underset{m}{S}} \underset{m}{a} = \overset{n}{\underset{m}{S}} \underset{2m-1}{a} + \overset{n}{\underset{m}{S}} \underset{2m}{a},$$

and $$\overset{2n-1}{\underset{m}{S}} \underset{m}{a} = \overset{n}{\underset{m}{S}} \underset{2m-1}{a} + \overset{n-1}{\underset{m}{S}} \underset{2m}{a}.$$

Art. 14. It will be sometimes found necessary to give to the indices subscript of the symbols of summation negative, as well positive integral values. In order to meet this necessity we may use $\overset{n,r}{\underset{m}{S}} . \underset{m}{a}$ to denote the

sum of all the terms arising from giving to m every integral value from n to r, zero being included when n is negative. We immediately see that

$$\overset{-n,\,r}{\underset{m}{S}}\, a_m = \overset{n}{\underset{m}{S}} .\, a_{-m} + \overset{r+1}{\underset{m}{S}}\, a_{m-1} \,;$$

and the generating function of $u_s = \overset{-\infty,\,\infty}{\underset{s}{S}} .\, u_s t^s$.

Art. 18. This most important Theorem may be proved more simply as follows:

$$\begin{aligned}
\overset{\infty}{\underset{m}{S}}\overset{\infty}{\underset{n}{S}} a_{m,n} &= \overset{\infty}{\underset{n}{S}}\overset{\infty}{\underset{m}{S}} a_{m,n} \\
&= \overset{\infty}{\underset{m}{S}} a_{m,1} + \overset{\infty}{\underset{m}{S}} a_{m,2} + \overset{\infty}{\underset{m}{S}} a_{m,3} + \text{\&c.} + \overset{\infty}{\underset{n}{S}} a_{m,n} + \text{\&c.} \\
&= a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1} + \text{\&c.} + a_{m,1} + \text{\&c.} \\
&\qquad + a_{1,2} + a_{2,2} + a_{3,2} + \text{\&c.} + a_{m-1,2} + \text{\&c.} \\
&\qquad\qquad + a_{1,3} + a_{2,3} + \text{\&c.} + a_{m-2,3} + \text{\&c.} \\
&\qquad\qquad\qquad + a_{1,4} + \text{\&c.} + a_{m-3,4} + \text{\&c.} \\
&\qquad\qquad\qquad\qquad + \text{\&c.} + \text{\&c.} + \text{\&c.} \\
&\qquad\qquad\qquad\qquad + \text{\&c.} + a_{m-n+1,n} + \text{\&c.} \\
&\qquad\qquad\qquad\qquad + \text{\&c.} + \text{\&c.} + \text{\&c.} \\
&\qquad\qquad\qquad\qquad\qquad + a_{1,m} + \text{\&c.} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \text{\&c.} \\
&= \overset{1}{\underset{n}{S}} a_{2-n,1} + \overset{2}{\underset{n}{S}} a_{3-n,n} + \overset{3}{\underset{n}{S}} a_{4-n,n} + \overset{4}{\underset{n}{S}} a_{5-n,n} + \text{\&c.} + \overset{m}{\underset{n}{S}} a_{m+1-n,n} + \text{\&c.} \\
&= \overset{\infty}{\underset{m}{S}} .\, \overset{m}{\underset{n}{S}} a_{m-n+1,n} \,.
\end{aligned}$$

The converse of this theorem may also be readily proved; viz.

$$\overset{\infty}{\underset{m}{S}}\overset{m}{\underset{n}{S}}a_{m,n} = \overset{\infty}{\underset{m}{S}}\overset{\infty}{\underset{n}{S}}a_{m+n-1,n}.$$

$$\overset{\infty}{\underset{m}{S}}\overset{m}{\underset{n}{S}}a_{m,n} = \overset{1}{\underset{n}{S}}a_{1,n} + \overset{2}{\underset{n}{S}}a_{2,n} + \overset{3}{\underset{n}{S}}a_{3,n} + \overset{4}{\underset{n}{S}}a_{4,n} + \&c.$$

$$= a_{1,1} + (a_{2,1} + a_{2,2}) + (a_{3,1} + a_{3,2} + a_{3,3}) + (a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4}) + \&c.$$

$$= a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1} + \&c.$$

$$+ a_{2,2} + a_{3,2} + a_{4,2} + a_{5,2} + \&c.$$

$$+ a_{3,3} + a_{4,3} + a_{5,3} + a_{6,3} + \&c.$$

$$+ a_{4,4} + a_{5,4} + a_{6,4} + a_{7,4} + \&c.$$

$$+ \&c.$$

$$= \overset{\infty}{\underset{m}{S}}a_{m,1} + \overset{\infty}{\underset{m}{S}}a_{m+1,2} + \overset{\infty}{\underset{m}{S}}a_{m+2,3} + \overset{\infty}{\underset{m}{S}}a_{m+3,4} + \&c.$$

$$= \overset{\infty}{\underset{n}{S}}\overset{\infty}{\underset{m}{S}}a_{m+n-1,n}.$$

$$= \overset{\infty}{\underset{m}{S}}\overset{\infty}{\underset{n}{S}}a_{m+n-1,n}.$$

The following theorem is analogous to that in the text. If r is not less than s, then

$$\overset{r}{\underset{m}{S}}\overset{s}{\underset{n}{S}}a_{m,n} = \overset{s}{\underset{m}{S}}\overset{m}{\underset{n}{S}}a_{m-n+1,n} + \overset{r-s}{\underset{m}{S}}.\overset{s}{\underset{n}{S}}a_{s+m-n+1,n} + \overset{s-1}{\underset{m}{S}}\overset{s-m}{\underset{n}{S}}a_{r-n+1,n+m}.$$

Art. 21. It is perhaps better to write $\overset{n}{\underset{m}{\{}}a\underset{n+1}{\{}$ instead of $\overset{n}{\underset{m}{\{}}a_m$. Among the numerous applications of this notation, that to continued fractions may be pointed out. Thus

$$\left\{\frac{a_m}{b_m + \{_{m+1}}\right._m^n = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \&c. + \cfrac{a_{n-1}}{b_{n-1} + \frac{a_n}{b_n}}}}.$$

Art. 29. I was not aware till a few weeks since that this notation is not original. It is just hinted at by Lacroix, *Traité du Cal. Diff. et Int.* Tom. II. p. 527.

Art. 31. This notation will be improved by writing $d_{x=a} . u$ instead of $d_{x,a} u$; and on the same principle $\{\phi(x)\}_{x=a} = \phi(a)$. Thus

$$\left\{\frac{x^n - 1}{x - 1}\right\}_{x=1} = \left\{\overset{n}{\underset{m}{S}} x^{m-1}\right\}_{x=1}$$

$$= n.$$

Art. 32. The necessity of adopting a significant notation to denote the limits of integration has been long felt. Fourier, who has been followed by Cauchy (*Memoire sur la Théorie des ondes*), proposed to adopt $\int_a^b u\, dx$.

Art. 33. Ex. 9. The very elegant principle on which the demonstration is founded, was demonstrated independently by Professor Schweins, and Mr. Coddington; the former in "*Theorie der Differenzen und Differentiale,*" Heidelberg, 1825, and the latter in "*An Introduction to the Differential Calculus on Algebraic Principles,*" Cambridge, 1825.

ERRATA.

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2 l. 9 *for* $a_n \,.\, a_{n-1} \,.\, a_{n-2} \,.\, a,$ *read* $a_n \,.\, a_{n-1} \,.\, a_{n-2} \ldots a_1$.

9 l. 2 *for* $S^n{}_m$ *read* $S_m{}^{n+1}$.

12 l. 4 *for* $(x-1)^t \,.\, S_r{}^n \, S_{r_1}{}^{r-1} \ldots S_{r_{t-1}}{}^{r_{t-1}-1} \,.\, 1$ *read* $(x-1)^t \,.\, S_r{}^n \, S_{r_1}{}^{r-1} \ldots S_{r_{t-1}}{}^{r_{t-2}-1} \,.\, 1$.

— l. 2 *for* $(x-1)^{t+1} \,.\, S_r{}^n \,.\, S_{r_1}{}^{r-1} \ldots S_{r_2}{}^{r_{t-1}-1} \, x^{r_t-1}$ *read* $(x-1)^{t+1} \,.\, S_r{}^n \, S_{r_1}{}^{r-1} \ldots S_{r_t}{}^{r_{t-1}-1} \, x^{r_t-1}$.

— l. 6 *for* (Art. 19.) *read* (Art. 11.)

— l. 11 *for* $= S_{r_1}{}^r \, S_{r_2}{}^{r_1} \, x^{r_3-1} \,.\, S_{r_3}{}^{r_1+1-r_2}$ *read* $= S_{r_1}{}^r \, S_{r_2}{}^{r_1} \, x^{r_2-1} \,.\, S_{r_3}{}^{r_1+1-r_2} \,.\, 1$ (Art. 17.).

— l. 12 *cancel* (Art. 16.).

— l. 16 *for* $= S_{r_t}{}^r \, x^{r_t-1} \, \underline{|r+t-1-r_1}_{\,t-1} \,.\, \frac{1}{\underline{|t-1}}$ *read* $= S_{r_1}{}^r \, x^{r_1-1} \, \underline{|r+t-1-r_1}_{\,t-1} \,.\, \frac{1}{\underline{|t-1}}$.

13 l. 3 *for* $\underline{|t+1-s}_{\,t}$ *read* $\underline{|t-1+s}_{\,t}$.

14 l. 7 *for* $\underline{|r}_{\,r-1}$ *read* $\underline{|r}_{\,s-1}$.

— l. 10 *for* $\frac{(-1)^{s-1}}{s-1}$ *read* $\frac{(-1)^{s-1}}{\underline{|s-1}}$.

15 l. 5 *for* $-n\left(\frac{m}{n}-r\right) m$ *read* $-n\left(\frac{m}{n}-r\right)\underline{|m}_{\,t}$.

16 l. 2 *for* $A_{s-1,\,b-1}$ *read* $A_{s-1,\,t-1}$.

17 l. 2 *for* $(1-1)^{r+1}$ *read* $(1-1)^{r-1}$.

— last line, *for* $\frac{n}{\underline{|r}}$ *read* $\frac{1}{\underline{|r}}$.

20 l. 5 *for* $\frac{(-1)^{s-1}}{s-1}$ *read* $\frac{(-1)^{s-1}}{\underline{|s-1}}$.

— l. 8 *for* $\underline{\left|\frac{m}{n}\right.} \,.\, \left(\frac{m}{n}-2\right)$ *read* $\frac{m}{n} \,.\, \left(\frac{m}{n}-2\right)$.

21 last line, *for* $\left(3-\frac{3}{\frac{m}{2}-2}\right)$ *read* $\left(3-\frac{3}{\frac{m}{n}-2}\right)$.

24 l. 8 *for* $\frac{\overline{v\,n|}^{\,r+1}}{\underline{|r+1}}$ *read* $\frac{\overline{u\,n|}^{\,r+1}}{\underline{|r+1}}$.

26 last line but 1, *for* (Art. 15.), *read* (Art. 16.).

28 l. 15 *for* $m=1$ *read* $n=1$.

29 last line but 1, *for* $\underline{|2^{2m}}$ *read* 2^{2m}.

III. *On the Disturbances of Pendulums and Balances; and on the Theory of Escapements.*

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Having lately had occasion to investigate the disturbance produced in the motion of a pendulum by a small external force, and having found by a very general investigation a result of great simplicity, I perceived that the usual theorems for the alteration in the time and extent of vibration produced by the difference between cycloidal and circular arcs, by the resistance of the air, by the friction at the point of suspension, &c. could be made to depend on it; and that these alterations could in fact be found with greater facility from this general theorem than from the independent and unconnected investigations. I found also that the same principles could be applied with great ease to that important practical subject, the escapements of clocks—a subject upon which I believe no distinct theory has ever yet been laid down. The investigations which have been the fruit of these considerations are now presented to the notice of this Society: the theory of escapements is by no means complete, but I hope it will be found that the principal points have been touched on, and that enough is said to enable any one else to pursue the subject as far as he may wish.

If l be the length of a pendulum vibrating in a cycloidal arc, x its distance at any instant from the position of rest, the equation of its motion is

$$\frac{d^2x}{dt^2} = -\frac{gx}{l}; \text{ or putting } n^2 = \frac{g}{l}, \frac{d^2x}{dt^2} + n^2x = 0.$$

The solution of this equation is

$$x = a \sin \overline{nt + b},$$

a and b being constants depending on the length of the arc and the time of passing the lowest position. The velocity at any time

$$= \frac{dx}{dt} = na \cos \overline{nt + b}.$$

Suppose now that, besides the force which varies as the distance, another very small accelerating force f acts on the ball of the pendulum in the direction in which x is measured positive:

$$\text{then } \frac{d^2x}{dt^2} = -\frac{gx}{l} + f, \text{ or } \frac{d^2x}{dt^2} + n^2x = f.$$

The solution of this equation may still be assumed $= a \cdot \sin \overline{nt+b}$, provided we consider a and b as functions of t. For whatever the solution may be, $a \cdot \sin \overline{nt + b}$ may be made equal to it by assuming either for a or for b a proper form. Since then a single assumption will satisfy this condition, and since we have two quantities whose forms are to be determined, it follows that we are at liberty to make another assumption. Let this be that the velocity shall be expressed by the same form as before, namely, $na \cdot \cos \overline{nt+b}$. The convenience of this assumption we shall soon discover.

Now since $x = a \,.\, \sin \overline{nt+b}$,

$$\therefore \frac{dx}{dt} = na \cos \overline{nt+b} + a \cos \overline{nt+b} \,.\, \frac{db}{dt} + \sin \overline{nt+b} \,.\, \frac{da}{dt}.$$

But the velocity or $\frac{dx}{dt}$ has been assumed $= na \cos \overline{nt+b}$: consequently,

$$a \cos \overline{nt+b} \,.\, \frac{db}{dt} + \sin \overline{nt+b} \,.\, \frac{da}{dt} = 0.$$

$$\text{And since } \frac{dx}{dt} = na \cos \overline{nt+b};$$

$$\therefore \frac{d^2x}{dt^2} = -n^2 a \sin \overline{nt+b} - na \sin \overline{nt+b} \,.\, \frac{db}{dt} + n \cos \overline{nt+b} \,.\, \frac{da}{dt}.$$

Substituting the values of x and $\frac{d^2x}{dt^2}$ in the original equation, we find

$$n \cos \overline{nt+b} \,.\, \frac{da}{dt} - na \sin \overline{nt+b} \,.\, \frac{db}{dt} = f.$$

Combining this with the equation above, or

$$\sin \overline{nt+b} \,.\, \frac{da}{dt} + a \cos \overline{nt+b} \,.\, \frac{db}{dt} = 0,$$

$$\text{we find } \frac{da}{dt} = \frac{f}{n} \,.\, \cos \overline{nt+b}, \quad \frac{db}{dt} = -\frac{f}{na} \,.\, \sin \overline{nt+b}.$$

If we could solve these two differential equations, we should have the complete determination of the motion.

In few cases, however, it is practicable to obtain an exact solution: and in all an approximation is sufficient for our purposes. This may be obtained by integrating the expressions

$$\frac{f}{n} \cos \overline{nt+b}, \quad \text{and} \quad -\frac{f}{na} \sin \overline{nt+b},$$

on the supposition that a and b are constant. As a and b are variable, this process is erroneous. But as their variation depends on f, the error depends on f^2, or on quantities of that order. Our approximation then will include all terms depending on the first power of f, and no more: an approximation sufficiently exact for all the cases to which we shall have to apply it.

Since the expression for the distance of the pendulum from its lowest position, and the expression for the velocity of the pendulum, are the same as those in an undisturbed cycloidal pendulum where a is the extent of the vibration, on each side, and $\frac{b}{n}$ the time which elapsed from the instant at which the pendulum passed its lowest position to the instant from which t is measured, it is plain that if the disturbing force ceased, the pendulum would move in the same manner: that is, the extent of its vibration would be a, and it would move as if it had passed its lowest point at the time $-\frac{b}{n}$, a and b having the values which they had at the instant when the force ceased to act. And generally, considering a and b as functions of t, the time of arriving at the lowest point will be determined by making

$$\sin \overline{nt + b} = 0,$$

and the time of reaching the highest point by making

$$\cos \overline{nt + b} = 0.$$

In order to find the alteration in the length of the arc of vibration which takes place in one oscillation, we must integrate $\frac{f}{n} \cos \overline{nt + b}$ through the limits of t corresponding to one oscillation; that is, from a value of t, which gives $nt + b = \alpha$, to the value of t, which gives $nt + b = \alpha + \pi$. Here α may be any thing

that we please: in some cases it will be convenient to take the integral from one extremity of the vibration to the other: in others it will be preferable to take it from the time at which the pendulum passes its lowest position to the time at which it again arrives there. In some cases it will be necessary to integrate for two vibrations.

To find the alteration in the length of time occupied by a vibration, produced in one oscillation, let B be the value of b at the first limit, and B' that at the second: and let T and T' be the times. Then the first time at which the pendulum passes its lowest point is found by making $nT + B = 0$: the second time is found by making $nT' + B' = \pi$. Hence

$$n(T' - T) + B' - B = \pi, \quad \text{and} \quad T' - T = \frac{\pi}{n} - \frac{B' - B}{n}.$$

But $T' - T =$ time occupied by one vibration: and

$$B' - B = \int_t \frac{db}{dt} = -\int_t \frac{f}{na} \sin \overline{nt + b},$$

between the proper limits. Consequently the time of oscillation is increased from

$$\frac{\pi}{n} \text{ to } \frac{\pi}{n} + \int_t \frac{f}{n^2 a} \sin \overline{nt + b}:$$

it is therefore increased by the proportional part

$$\frac{1}{\pi n a} \int_t f \,.\, \sin \overline{nt + b}.$$

Recapitulating, then, we have

$$\text{increase of arc of semi-vibration} = \frac{1}{n} \int_t f \,.\, \cos \overline{nt + b},$$

$$\text{proportionate increase of time of vibration} = \frac{1}{\pi n a} \int_t f \,.\, \sin \overline{nt + b}.$$

If the circumstances were such that it was necessary to integrate through two vibrations, we should have

$$\text{proportionate increase of time of vibration} = \frac{1}{2\,\pi n a}\int_t f\,.\,\sin \overline{nt+b}.$$

These formulæ are convenient when the disturbing forces can be expressed in terms of t. If, however, they are expressed in terms of x (as is the case particularly in clock escapements),

$$\text{since } \frac{da}{dx} = \frac{da}{dt}\cdot\frac{dt}{dx} = \frac{\frac{da}{dt}}{na\cos\overline{nt+b}} = \frac{f}{n^2 a},$$

$$\text{and } \frac{db}{dx} = \frac{\frac{db}{dt}}{na\cos\overline{nt+b}} = -\frac{f}{n^2 a^2}\cdot\frac{\sin\overline{nt+b}}{\cos\overline{nt+b}} = -\frac{f}{n^2 a^2}\cdot\frac{x}{\sqrt{a^2-x^2}},$$

we have

$$\text{increase of arc of semi-vibration} = \frac{1}{n^2 a}\int_x f,$$

$$\text{proportionate increase of time of vibration} = \frac{1}{\pi n^2 a^2}\int_x \frac{fx}{\sqrt{a^2-x^2}}.$$

EXAMPLE 1. Instead of vibrating in a cycloid, the pendulum vibrates in a circle. Here the force

$$= -g\,.\,\sin\frac{x}{l} = -g\left(\frac{x}{l} - \frac{x^3}{6l^3}\right) \text{ nearly} = -\frac{gx}{l} + \frac{g}{6l^3}x^3;$$

$$\therefore f = \frac{g}{6l^3}x^3 = \frac{ga^3}{6l^3}\sin^3\overline{nt+b}:$$

and the proportionate increase of the time of vibration

$$= \frac{ga^2}{6\pi n l^3}\int_t \sin^4.\overline{nt+b}.$$

$$\text{But } \int_t \sin^4 \overline{nt+b} \text{ from } nt+b=0 \text{ to } nt+b=\pi \text{ is } \frac{3}{8}\cdot\frac{\pi}{n};$$

therefore the proportionate increase of the time of vibration

$$= \frac{g a^2}{16 n^2 l^3} = \left(\text{since } n^2 = \frac{g}{l}\right) \frac{a^2}{16 l^2}.$$

Ex. 2. The friction at the point of suspension is constant. Here $f = -c$: and it will be convenient to take the integrals during that time in which the friction acts in the same direction, that is, from the beginning of a vibration to its end, or from

$$nt + b = -\frac{\pi}{2}, \text{ to } nt + b = \frac{\pi}{2}.$$

Hence the increase of the arc of semi-vibration

$$= -\frac{c}{n}\int_t \cos \overline{nt+b}:$$

which from $nt+b = -\frac{\pi}{2}$ to $nt+b = \frac{\pi}{2}$ gives the increase $= -\frac{2c}{n^2}$.
The proportionate increase of the time of vibration

$$= -\frac{c}{\pi n a}\int_t \sin \overline{nt+b}:$$

which between the same limits is 0.

Ex. 3. The resistance varies as the m^{th} power of the velocity, or $= kv^m$, m being any whole number.

$$\text{Here } f = -kn^m a^m . \cos^m . \overline{nt+b}.$$

Hence the increase of the arc of semi-vibration

$$= -kn^{m-1} . a^m . \int_t \cos^{m+1} . \overline{nt+b},$$

to be taken between the same limits as in the last example.

This gives $\int_t \cos^{m+1} \overline{nt+b} = \dfrac{m \,.\, \overline{m-2} \ldots 1}{\overline{m+1} \,.\, \overline{m-1} \ldots 2} \cdot \dfrac{\pi}{n}$ when m is odd,

$$\text{and } = \frac{m \,.\, \overline{m-2} \ldots 2}{\overline{m+1} \,.\, \overline{m-1} \ldots 2} \cdot \frac{2}{n} \text{ when } m \text{ is even.}$$

Thus the decrease of the arc

$$= \frac{m \,.\, \overline{m-2} \ldots 1}{\overline{m+1} \,.\, \overline{m-1} \ldots 2} k\pi n^{m-2} \,.\, a^m \ (m \text{ odd}),$$

$$\text{or } = \frac{m \,.\, \overline{m-2} \ldots 2}{\overline{m+1} \,.\, \overline{m-1} \ldots 3} 2k \,.\, n^{m-2} \,.\, a^m \ (m \text{ even}).$$

$$\text{When } m=2 \text{ this } = \frac{4}{3} ka^2.$$

The proportionate increase of the time of vibration

$$= -\frac{k}{\pi} n^{m-1} \,.\, a^{m-1} \,.\, \int_t \cos^m \overline{nt+b} \,.\, \sin \overline{nt+b},$$

which between the same limits = 0, whether m be whole or fractional.

Ex. 4. The resistance is expressed by any function of the velocity. Here $f = -\phi(v)$, and the increase of the arc

$$= -\frac{1}{n} \int_t \phi(v) \,.\, \cos \overline{nt+b}$$

to be taken from

$$nt+b = -\frac{\pi}{2} \text{ to } nt+b = \frac{\pi}{2}.$$

$$\text{Since } v = na \cos \overline{nt+b}, \quad \frac{dv}{dt} = -n^2 a \,.\, \sin \overline{nt+b},$$

and the increase of the arc

$$= \frac{1}{n^3 a} \int_v \phi(v) \frac{\cos \overline{nt+b}}{\sin \overline{nt+b}} = \frac{1}{n^3 a} \int_v \frac{v\,\phi(v)}{\sqrt{n^2a^2 - v^2}},$$

the integral being taken from $v=0$ to $v=0$ again. But it must be observed that from $v=0$ to $v=na$, the radical must be taken with a negative sign, because $\sin \overline{nt+b}$ is then negative. The increase of the arc is, therefore, the sum of

$$-\frac{1}{n^3a}\int_v \frac{v\phi(v)}{\sqrt{n^2a^2-v^2}} \begin{Bmatrix} v=0 \\ v=na \end{Bmatrix} \text{ and } \frac{1}{n^3a}\int_v \frac{v\phi(v)}{\sqrt{n^2a^2-v^2}} \begin{Bmatrix} v=na \\ v=0 \end{Bmatrix}:$$

that is, the decrease of the arc is

$$\frac{2}{n^3a}\int_v \frac{v\phi(v)}{\sqrt{n^2a^2-v^2}} \begin{Bmatrix} v=0 \\ v=na \end{Bmatrix}.$$

The proportional increase of the time of vibration is

$$\frac{-1}{\pi na}\int_t \phi(v)\,.\,\sin\overline{nt+b} = \frac{1}{\pi n^3a^2}\int_t \phi(v)\,.\,\frac{dv}{dt} = \frac{1}{\pi n^3a^2}\int_v \phi(v) = \frac{1}{\pi n^3a^2}\psi(v).$$

This taken between the limits $\begin{Bmatrix} v=0 \\ v=0 \end{Bmatrix}$ is in all cases $=0$. A resistance, therefore, which is constant, or which depends on the velocity, does not alter the time of vibration.

Ex. 5. The resistance is that produced by a current of air moving in the plane of vibration with a velocity V greater than the greatest velocity of the pendulum; and varies as the square of their relative velocity. In this case, when the pendulum moves in the direction of the current

$$\phi(v) = -k(V-v)^2,$$

and when it moves in the opposite direction,

$$\phi(v) = k(V+v)^2.$$

By the formula above we find that when the pendulum moves in the direction of the current, the arc is increased by

$$k\left(\frac{2V^2}{n^2} - \frac{Va\pi}{n} + \frac{4a^2}{3}\right),$$

and when it returns, the arc is diminished by

$$k\left(\frac{2V^2}{n^2}+\frac{Va\pi}{n}+\frac{4a^2}{3}\right).$$

The diminution in two vibrations is, therefore, $k.\frac{2Va\pi}{n}$. The time of vibration is unaltered.

Ex. 6. The resistance is that produced by a current of air whose velocity is not equal to the greatest velocity of the pendulum. Here, when the pendulum moves in the direction of the current $\phi(v)=-k(V-v)^2$ when v is $< V$, and $\phi(v)=k(v-V)^2$ when v is $>V$. By the formula above, the increase of the arc is

$$\frac{k}{n^3a}\left\{2V^2na-4n^2a^2\sqrt{n^2a^2-V^2}+\frac{4}{3}(n^2a^2-V^2)^{\frac{3}{2}}-4Vn^2a^2\left(\sin^{-1}\frac{V}{na}-\frac{\pi}{4}\right)\right\}.$$

The time is not altered. The motion in the opposite direction is the same as in the last Example.

Ex. 7. The force F acts through a very small space s at the distance c from the lowest point. For the increase of the arc we must take

$$\frac{1}{n^2a}\int_x F\begin{Bmatrix}x=c\\x=c+s\end{Bmatrix}.$$

This is plainly $=\frac{Fs}{n^2a}$.

The proportionate increase of the time of vibration

$$=\frac{1}{\pi n^2a^2}\int_x\frac{Fx}{\sqrt{a^2-x^2}}\begin{Bmatrix}x=c\\x=c+s\end{Bmatrix};$$

if the general value of the integral be $\phi(x)$, the value between these limits will be $\phi(c+s)-\phi(c)=\phi'(c).s$ nearly

$$=\frac{Fs}{\pi n^2a^2}\cdot\frac{c}{\sqrt{a^2-c^2}}.$$

If then an impulse be given when the pendulum is at its lowest point, c is o, and the time of vibration is unaffected.

Ex. 8. A force f which is equal at equal distances from the lowest point on both sides accelerates the pendulum. By the general formulæ it will be found that the action on both sides of the lowest point tends to increase the arc: but that the action before reaching the lowest point tends to diminish the time of oscillation, and that after it to increase the time, and that on the whole the time of oscillation is not altered.

Ex. 9. A force M which is equal at equal distances retards the pendulum as it ascends from the distance c to its highest point, and accelerates it as it descends to the same place. Upon taking $\frac{-1}{n^2 a} \int_x M$ from $x = c$ till x again $= c$, we find that the length of the arc is not altered. In rising the time of vibration is increased by

$$-\frac{1}{n^3 a^3} \int_x \frac{Mx}{\sqrt{a^2 - x^2}} \begin{Bmatrix} x = c \\ x = a \end{Bmatrix}.$$

To find the effect produced as the pendulum descends we must remark that $\sqrt{a^2 - x^2}$ was introduced as equal to $a \cos \overline{nt + b}$, which then becomes negative; and the radical must therefore be taken with a negative sign. We must, therefore, take

$$-\frac{1}{n^3 a^2} \int_x \cdot \frac{Mx}{-\sqrt{a^2 - x^2}} \begin{Bmatrix} x = a \\ x = c \end{Bmatrix}, \quad \text{or} \quad -\frac{1}{n^3 a^2} \int_x \frac{Mx}{\sqrt{a^2 - x^2}} \begin{Bmatrix} x = c \\ x = a \end{Bmatrix}.$$

The whole decrement in the time is, therefore,

$$\frac{2}{n^3 a^2} \int_x \cdot \frac{Mx}{\sqrt{a^2 - x^2}} \begin{Bmatrix} x = c \\ x = a \end{Bmatrix}.$$

A force of this kind then does not alter the arc of vibration, but tends during the whole of its action to diminish the time.

Since the theory is applicable to every case in which a pendulum is acted on by small forces, it can be applied to determine the effect produced on the motion of the pendulum of a clock, or the balance of a watch, by the machinery which serves to maintain that motion. After describing generally the manner in which the weight by the intermediation of the wheel-work acts on the pendulum, and stating the principles to be observed in the construction of escapements which follow from the investigations above, we shall proceed to examine the escapements which are most in use.

If a pendulum vibrates uninfluenced by any external forces except that of gravity, the resistance of the air and the friction at the point of suspension reduce gradually the extent of vibration. But this diminution goes on very slowly. I have observed a pendulum suspended on knife edges vibrate more than seven hours before its arc was reduced from two degrees to $\frac{1}{5}$th of a degree. In order to maintain vibrations of the same or nearly the same length (which for clocks is indispensible) a force must act on the pendulum: this force is generally given by the action of a tooth of the seconds wheel on the inclined surfaces of small arms or pallets carried by the pendulum: and the whole apparatus is called the escapement. It is necessary, therefore, in the theory of escapements, to consider the motion of the pendulum when, besides the force arising from its own weight, it is acted on by the resistance of the air, &c. and by the force impressed by the machinery. The fact stated above shews that the first of these forces, and consequently the second, are so small that our approximate theory is abundantly sufficient for this investigation.

Now it appears from Examples 2, 3, 4, 5, and 6, that the friction and the resistance of the air do not affect the time of vibration. The maintaining force, therefore, must be impressed

in such a manner as not to alter the time of vibration. With this construction a compensated pendulum moving in a cycloidal arc would be isochronous. But the pendulums of clocks swing in circular arcs, and it might therefore appear desirable to make the escapement in such a manner as to correct the difference between the circular and the cycloidal vibration. In this manner the oscillations would always be isochronous, whatever variations the maintaining power and the length of the arc of oscillation might undergo.

Upon a more accurate examination, however, I believe it will be found much better to lay aside all thoughts of this artifice. The law of the air's resistance when the velocity is so small as that of a pendulum is not known. By observations on detached pendulums of Captain Kater's construction, I have found that it differs very much from that of the square of the velocity. Whether it would be better represented by the simple velocity, or by the square of the velocity increased by a constant, I do not know. Besides this, it is almost impossible to ascertain the effect of the friction upon the pallets, or its proportion to the resistance of the air, &c. Since then we cannot express, in a known function of the length of the arc, the diminution of the arc at each vibration, or the quantity which the maintaining force must increase it at each vibration, we cannot find what the force must be to maintain a given extent of vibration, and, therefore, we cannot find in what manner it must be applied that its effect on the time of vibration may exactly counteract that of the difference between the circular and cycloidal arcs. It is possible also in pendulums with a spring suspension, to make the vibrations very nearly isochronous in different arcs; and in pendulums with knife edge suspensions, it is easy to apply a construction which will have the same effect. I shall, therefore, consider it as the object in the construction of escapements to make them in such a manner that they do not at all affect the time of vibration.

The escapements of clocks in general use may be divided into the three following classes: recoil escapements, dead-beat escapements: and the escapements in which the action of the wheels raises a small weight which by its descent accelerates the pendulum. The last may be called, from the name of their first proposer, Cumming's escapements.

I shall first observe that the friction which in the recoil and dead-beat escapements takes place during the whole vibration does not appear to affect the time of vibration. This friction may be separated into two parts: that which is properly called friction, arising from the rubbing of two bodies, and that which arises from the viscidity of the oil. The former of these is generally considered to be constant, and the latter to vary nearly as the velocity. Consequently, by Examples 2 and 3, they do not alter the time of oscillation. It is undoubtedly important that friction should be avoided if possible, as a smaller maintaining power is then required, and the irregularities which it may occasion in the pendulum's motion are proportionally diminished. In the dead-beat escapement this friction is interrupted during the time in which the wheel is acting on the pallets. We may, however, suppose a retarding force to act during this time, provided we add to the maintaining power an equal force. In Cumming's escapement the friction is nothing.

In the recoil escapement, soon after the pendulum has passed its lowest position a force begins to retard it till it reaches the extremity of its vibration: then (acting still in the same direction) it accelerates it till it has again passed the lowest point by the same distance as before: then another retarding force commences its action, &c. We may then consider the action of the force as divided into two parts, of which one retards the ascent of the pendulum and accelerates its descent, and the other accelerates the pendulum a little before and a little after it has reached its lowest point. The former of these, by Example 9, has no effect

at all in maintaining the arc of vibration, but always diminishes the time. The latter, by Example 8, if it be equal on both sides of the lowest point has no effect on the time of vibration, but increases the arc of vibration if there be no resistance, or maintains it if there be resistance. The former of these, then, is of no use whatever, and is prejudicial as affecting the rate of the clock: the latter does not affect the rate, and fulfils the office of an escapement by maintaining the motion of the pendulum. Thus we see that the principle of this escapement is radically bad. The force during the greater part of its action is disturbing the rate of the clock without maintaining the motion of the pendulum.

If the pallets have such a form that the force is constant and $= F$, we find by Example 9, that the time of vibration is diminished by

$$\frac{2}{n^3 a^2} \int_x \frac{Fx}{\sqrt{a^2 - x^2}} \begin{Bmatrix} x = c \\ x = a \end{Bmatrix} = \frac{2F}{n^3 a^2} \sqrt{a^2 - c^2}.$$

The differential coefficient of this quantity with respect to a is

$$\frac{-2F}{n^3} \cdot \frac{a^2 - 2c^2}{a^3 \sqrt{a^2 - c^2}}.$$

Hence it appears that the vibrations are quicker than they would be without the maintaining force: but that if from a diminution of friction, &c. the arc be increased while the maintaining force remains the same, the vibrations are slower. If while the arc remains the same the maintaining force be increased, the vibrations are quicker.

If the resistance varied as the square of the velocity, the diminution of the arc from that cause (see Example 3.) would be

$\frac{4k}{3}a^2$: and the increase caused by the force F (found in the same manner) would be $\frac{2Fc}{n^2 a}$. Hence $\frac{4k}{3}a^2 = \frac{2Fc}{n^2 a}$, whence

$$F = \frac{2kn^2}{3} \cdot \frac{a^3}{c},$$

and the diminution in the time of vibration

$$= \frac{4k}{3\pi} \cdot \frac{a\sqrt{a^2-c^2}}{c}.$$

The differential coefficient of this with respect to a is

$$\frac{4k}{3\pi} \cdot \frac{2a^2-c^2}{c\sqrt{a^2-c^2}}.$$

Here then the vibrations are performed quicker in the large arcs than in the small ones.

If the force, instead of being constant from c to a and from a to c, varied directly as the distance, putting, in Example 9, $M = ex$, the time is diminished by

$$\frac{2}{n^3 a^2} \int_x \frac{ex^2}{\sqrt{a^2-x^2}} \begin{Bmatrix} x = c \\ x = a \end{Bmatrix},$$

$$\text{or by } \frac{e}{n^3 a^2}\left\{c\sqrt{a^2-c^2} + a^2\left(\frac{\pi}{2} - \sin^{-1}\frac{c}{a}\right)\right\} = \frac{e}{n^3 a^2}\left\{a^2\frac{\pi}{2} - \frac{c^3}{2a^2}\right\},$$

nearly, when c is small: which is almost independent of a. Here then an alteration in the arc of vibration would scarcely affect the clock's rate: but an alteration in the maintaining power would affect it greatly.

We may thus investigate the possibility of a law of resistance that will make the vibrations isochronous, however the maintaining power may vary. Suppose the force on the pallets constant in

the same vibration: then, as it is required that the acceleration produced by the maintaining power shall be invariable, we must have

$$\frac{2F}{\pi n^2} \cdot \frac{\sqrt{a^2-c^2}}{a^2} = \text{a constant} = \frac{2C}{\pi n^2}, \text{ or } F = \frac{Ca^2}{\sqrt{a^2-c^2}}.$$

The increase of the arc in consequence will be

$$\frac{2Cac}{n^2\sqrt{a^2-c^2}} = \frac{2Cc}{n^2}\left(1+\frac{c^2}{2a^2}\right) \text{ nearly.}$$

No resistance expressed in positive powers of the velocity will give a diminution of the arc of this form, and therefore it is impossible to make the vibrations isochronous.

Besides the forces already considered, there is the impact on the pallet which takes place at the beat. As this depends on the weight of the wheels, &c., it is impossible to measure it, but we can discover the nature of the effect which it will produce. It may be represented by a force $-G$, acting through a very small space h at the distance c. It will therefore diminish the time of vibration by

$$\frac{1}{\pi n^2 a^2} \cdot \frac{Ghc}{\sqrt{a^2-c^2}}.$$

This evidently diminishes as a is increased, and therefore this force produces the same effect as the others.

We have now sufficient data to enable us to form a judgment of the merits of this escapement. It appears that the maintaining power will always enter into the expression for the time of vibration: consequently, any obstruction or friction of the wheel-work will affect the rate of the clock. The arc also generally enters, and therefore any alteration of the extent of vibration

will alter the rate. And with the usual form of the pallets no law of resistance can be found which will make the vibrations isochronous.

In Cumming's escapements, the action of the wheels raises a weight through a small space. It is then carried up by the pendulum, and descends with the pendulum, till the latter has arrived at its lowest point. This case, then, is almost exactly the same as the last, with this exception, that the force which acts on the pendulum is independent of the irregularities in the force transmitted through the wheel-work. If, however, the length of the arc of vibration undergo any change, the time of vibration will be changed. The principle of this construction is, therefore, almost as bad as that of the former.

We now come to the dead-beat escapement. Here the wheel acts on the pallet for a small space near the middle of the vibration: and during the remainder of the vibration it has no effect except in producing a slight friction. The impact also at the beat does not tend to accelerate or retard the pendulum. Neglecting then the consideration of the friction, we have a constant force F, which begins to act when $x = -c$, and ceases when $x = c'$. Taking

$$\frac{1}{\pi n^2 a^2} \int_x \frac{Fx}{\sqrt{a^2 - x^2}}$$

between those limits, we have for the proportional increase of time,

$$\frac{F}{\pi n^2 a^2} \left(\sqrt{a^2 - c^2} - \sqrt{a^2 - c'^2}\right) = \frac{F}{\pi n^2 a^2} \cdot \frac{c'^2 - c^2}{\sqrt{a^2 - c^2} + \sqrt{a^2 - c'^2}}$$

$$= \frac{F}{2\pi n^2 a^3} \cdot \overline{c' + c} \cdot \overline{c' - c}, \text{ nearly.}$$

This is a quantity extremely minute. For c and c' are generally small compared with a, and $c' - c$ may be made almost as small

as we please; consequently $\overline{c'+c}\,.\,\overline{c'-c}$ is very small. If c and c' were equal, the effect on the time would be absolutely nothing. With this escapement, therefore, the effect of the maintaining power on the rate of the pendulum may be made as small as we please.

It cannot, however, be made absolutely nothing. For the wheel must be so adapted to the pallets, that when it is disengaged from one it may strike the other, not on the acting surface, but a little above it. That is, the instant of disengagement from a pallet must follow the instant at which the pendulum is in its middle position by a rather longer time than that by which the instant of beginning to act preceded it. Therefore, c' must be rather greater than c. But the difference may be made so small that the effect on the clock's rate shall be almost insensible. This escapement, therefore, approaches very nearly to absolute perfection: and in this respect theory and practice are in exact agreement.

The impact at the beat indirectly affects the time of vibration by producing, for a very short time, a considerable friction. This cannot be estimated: but it will easily be seen that, as it takes place after the pendulum has passed its lowest point, its effect is to diminish the time of vibration.

As the force of the spiral spring on a watch balance is (in the best springs) proportional to the angular distance of the balance from its position of rest, the same equations which apply to the motion of a pendulum will apply also to the motion of a balance. The comparative merits of watch escapements can, therefore, be determined in the same manner as those of clock escapements. The common crown wheel and verge escapement is precisely similar in its action to the recoil escapement of clocks, and has exactly the same defects. Mudge's detached escapement for watches is exactly similar in principle to Cumming's

for clocks, with this exception, that the force varies as the distance from the middle position. Putting ex for this force, and observing that it retards the balance from the distance c to a, and then accelerates it from a to o, we find for the proportionate increase in the time

$$\frac{e}{\pi n^2 a^2}\left(-a^2\frac{\pi}{2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{c}{a}-\frac{c\sqrt{a^2-c^2}}{2}\right):$$

which, if c be not very large, is nearly equal to

$$-\frac{e}{2n^2}-\frac{ec^3}{3\pi n^2 a^3}.$$

The first of these terms is large, but invariable: the second, which is variable, is not so small as at first sight it appears. For it will be found that the arc of semi-vibration is increased by $\frac{ec^2}{2n^2 a}$: and this must be equal to A, the quantity by which the arc of semi-vibration is diminished by friction, &c. The product ec^2 is, therefore, of the same order as A, and, therefore, the term $\frac{ec^3}{3\pi n^2 a^3}$ is of the order cA, which is not to be neglected. The only variable quantity in this term is a: if from an alteration in the quantity of friction, &c., the length of the arc be increased, the time of vibration will be increased, but c will be a multiplier of the expression for the increase. This, therefore, so far as the theory is concerned, is a pretty good construction. Graham's cylinder escapement, and Mudge's lever escapement (now extensively used under the name of *the detached lever*), possess almost exactly the same properties as the dead-beat escapement of clocks, and are, therefore, very good. The duplex escapement is an instance of a class differing from all those which we have mentioned in one important respect — the maintaining power acts on the balance only once in two vibrations. For the rest, its action (like that

in the dead-beat escapement) lasts for a very short time. It is, therefore, in the power of the artist to construct it in such a manner that the action shall take place equally before and after the time at which the balance reaches the middle of its vibration. In this manner the quantities c and c', in the investigation for the dead-beat escapement, would be equal, and the effect of the maintaining power on the rate of the chronometer would be absolutely nothing. It appears to be owing to this that the duplex escapement is found to be so good. The only point in which the detached escapements of Arnold and Earnshaw appear to be superior (which is, however, a point of importance) is the almost perfect absence of friction. As the wheel touches the balance only once in two vibrations, the latter may be so adjusted that the time of vibration shall be perfectly independent of the maintaining power. If the slight resistance offered by the springs be taken into account, the same is true. When to this consideration we add that the motion of the balance is not clogged by any friction, except that of its own pivots and spring, it does not appear possible to form an escapement more perfect in theory than these. The reasons which determine the form of the teeth of the wheel in these two escapements are entirely practical: provided the action takes place during a small part only of the vibration, it is indifferent whether the force be uniform or not.

We have seen that the rank which is assigned to the different escapements by theory is precisely the same as that which is given by experience: and this circumstance seems to justify us in the presumption that nothing important is omitted in the view that we have taken of this theory. Perhaps then I may be allowed to suggest, on mere theoretical considerations, a form for the escapement of clocks: similar in its principles to the best detached escapements of chronometers, and apparently likely to possess the same advantages. In fig. 1, A is the wheel whose axis

carries the seconds' hand: it is represented in the figure with 60 pins perpendicular to its plane: instead of a wheel with pins, a crown wheel can be used. *B* is the crutch for the pendulum, in its position of rest. *C* the pallet, carried by the arm *CD*: in the position of rest it will be between two pins of the wheel. *E* a small pin carried by the arm *CD*. *FG* a spring fixed at *F*: flexible towards *F*, and stiff in the remaining part: it carries a tooth *H* concealed in fig. 1, by the rim of the wheel, but shewn in fig. 2: this tooth stops the wheel *A* by catching one of its pins. *KL* a very weak spring attached at *K* to the under side of *FG*: the part at *L* projects beyond the rest, that it may be touched by the pin *E*. The action of this escapement is very simple. When the pendulum moves so that *C* approaches the center of the wheel, the pin *E* touches *L*, and (as the spring *KL* is very weak) passes it without any sensible retardation. When it returns, immediately before the middle of its vibration, the same pin touches the under side of *L*, and as *L* cannot yield without raising *FG*, it lifts the tooth *H* from the pin of the wheel which it held, and the wheel thus set at liberty immediately acts on the pallet *C*. Before it has finished its action, the pin *F* has let go the spring *L*, and *H* has descended so as to catch another tooth as soon as the action shall have ceased. The pendulum, it is plain, receives but one impulse in two vibrations, and, therefore, for a seconds' beat, a half second pendulum would be necessary.

To investigate the effect of these actions on the time of vibration, omitting nothing, let z be the distance of the pendulum from its middle point when the action on the pallet begins, $z+c$ that when it ends, $z-e$ the mean distance at which the resistance of the spring *FG* takes place, and $z-k$ that at which the resistance of *KL* takes place as the pendulum returns. Let the action on the pallet be supposed constant and $= F$: and for the springs,

let the product of the force by the space through which they continue their action be G and K respectively. Here c, e, and k depend on the construction of the clock, and z on the situation of the pallet when the pendulum is at the middle of its vibration, that is, on the position of the clock. Observing that when the pendulum is going, we must take

$$\frac{1}{2\pi n^2 a^2}\int_x \frac{fx}{\sqrt{a^2-x^2}},$$

and when it is returning,

$$\frac{-1}{2\pi n^2 a^2}\int_x \frac{fx}{\sqrt{a^2-x^2}},$$

for the proportionate increase, and observing that the first and third forces are positive, and the second negative, we find for the proportionate increase

$$\frac{1}{2\pi n^2 a^2}\left\{F(\sqrt{a^2-z^2}-\sqrt{a^2-(z+c)^2})-\frac{G(z-e)}{\sqrt{a^2-(z-e)^2}}+\frac{K(z-k)}{\sqrt{a^2-(z-k)^2}}\right\},$$

or approximately,

$$\frac{1}{2\pi n^2 a^2}\left\{F.c.\overline{z+\frac{c}{2}}-G.\overline{z-e}+K\overline{z-k}\right\},$$

This will be nothing when

$$z=-\frac{c}{2}\times\frac{2Fc^2+4Ge-4Kk}{2Fc^2-2Gc+2Kc}=-\frac{c}{2}-\text{a small quantity.}$$

It appears, therefore, that the time of action on the pallet before the pendulum reaches the middle point must exceed, by a very small quantity, that after it has passed the middle point, in order that the time of vibration may be independent of the maintaining force. If the action began before this time, there would be a term of the form $-\frac{AF+B}{a^3}$, or $-\frac{AF}{a^3}$ nearly, in the time of vibration: that is, the clock's rate would be made quicker, but less for large

arcs than for small (supposing F to be unchanged): if the action began after it, the contrary effect would take place. If we take for F a force sufficient to counteract the resistance of the air supposed to be as the square of the velocity; since the diminution of the arc from that cause $= ma^2$, and the increase from the action of $F = \frac{Fc}{n^2 a}$, we must have $\frac{Fc}{n^2 a} = ma^2$, or $F = \frac{mn^2 a^3}{c}$; $\therefore \frac{AF}{a^3}$ is independent of a, and the clock's rate would be the same under all variations of the maintaining power. If the resistance be supposed to follow any other law, the clock's rate will not be independent of a, except the action takes place as is above described.

The advantages which this escapement seems to offer are as follows:

1st. It would not require greater delicacy of workmanship, perhaps less, than the common dead-beat.

2d. It possesses in theory all the advantages of the detached escapement in watches.

3d. The clock would never be out of beat.

Whether the shortness of the pendulum for a given rate of beat would be any disadvantage, I do not know. The construction would, however, I apprehend, be very convenient for a portable clock. It is evident that by a repetition of this construction, the clock might be made to beat at every vibration: but several advantages would be lost by this complication.

G. B. AIRY.

TRINITY COLLEGE,
Nov. 10, 1827.

Fig. 1.

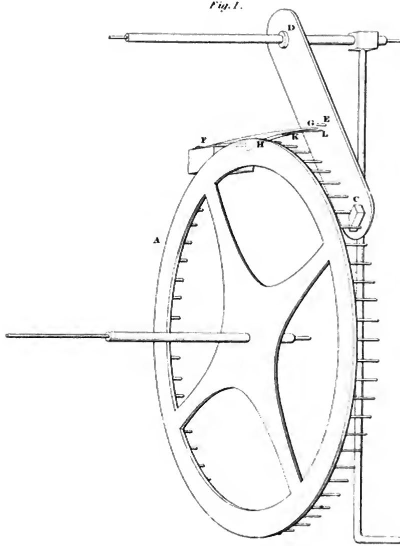

Fig. 2.

Fig. 3.

E. Turrell. sc.

IV. *On the Pressure produced on a flat Plate when opposed to a Stream of Air issuing from an Orifice in a plane Surface.*

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[Read *April* 21, 1828.]

The singular and apparently paradoxical result obtained by opposing a flat plate to a current of elastic fluid issuing from an orifice in a plane surface, has lately excited considerable interest, especially on the Continent, where it was first brought into notice. In its simplest form the experiment consists in this. If we blow through a tube, the aperture of which terminates in a flat plate, and apply a circular disc of card or any other convenient material to the aperture, we find that as long as the blast is continued, the disc is attracted to the plate instead of being repelled, as might naturally have been expected. Some pins must be fixed into the plate, to prevent the disc from slipping off sideways.

This appears to have been first discovered, apparently by accident, at the iron-works of Fourchambault, where one of their forge-bellows opened in a flat wall, and it was found that a board presented to the blast was sucked up against the wall. It was there exhibited to Messrs. Thenard and Clement Desormes, in October 1826, and shortly afterwards a paper appeared in the *Bulletin Universel,* in which the latter gentleman considered a similar

phenomenon with respect to the escape of steam under high pressure, and the danger of failure to which the common safety-valves of steam-boilers were exposed, by this singular fact. M. Hachette then succeeded in simplifying the form of the experiment, so that it might be performed by a pair of common bellows, or a stream of air from the mouth. He also produced the same effects by using a stream of water instead of air. (The particulars may be found, *Bull. Univ.* E. VII. pp. 41. 104. *Ann. de Chimie,* 1827, T. XXXV. p. 34, and T. XXXVI. p. 69. *Quarterly Journal,* 1827, Vol. I. p. 472, and Vol. II. p. 193. Some similar phenomena may, however, be seen in *Young's Nat. Phil.* Vol. I. pp. 298, 778.)

My object in the present Memoir is to detail some experiments which were undertaken for the purpose of examining some of the laws of this phenomenon more minutely than has hitherto been done.

In order to put it into a shape more convenient for investigation, some tubes of different diameters, terminating in flat plates, were connected to the wind-chest of an organ capable of furnishing a regular blast of any pressure not exceeding six inches of water, and a balance of light wood about six feet long, together with a number of discs of tin of various diameters, which could be attached by means of a screw to one of its extremities, being provided; then, by adding weights to the other extremity, and counterbalancing these by placing known weights on the centers of the discs, the effects of varying the orifice, pressure, &c. could be measured. The balance was made of considerable length, that the parallelism of the discs might not be sensibly affected by its motion.

Let *CBD*, Fig. 1, be a section of the lower plate provided with its tube *AB*, through which a constant blast is maintained. Bring the upper disc *GH* gradually down to *CD*, preserving its

parallelism with the lower plate, and keeping its center perpendicularly over the center of the orifice. It will be at first violently repelled by a force which will be found to increase till the disc reaches a point *k*, thence the force diminishes to a point *l*, where the disc appears in a state of unstable equilibrium. Bringing it still lower it will be attracted by a force which increases, reaches a maximum at *m*, and diminishes till the disc is placed in stable equilibrium at *n*, and will be repelled if pressed still further down.

TABLE I.

Diameter of disc.................	.8	1	1.5	2	2.5	3	4
Distance of stable equilibrium...	.015	.015	.02	.025	.025	.025	.025
Distance of unstable equilibrium.	.07	.12	.23	.3	.36	.43	.6

This table shews the distance of these points of stable and unstable equilibrium from the lower plate, in the case of an orifice .25 in diameter, and a pressure of 6 inches. *The measures throughout this paper must be understood of inches, unless otherwise expressed.* The next table shews the weight in grains required to pull discs of various diameters off the lower plate. It must be observed, however, that it is extremely difficult to ascertain the exact weight required to do this, and this difficulty is increased by a certain tremulous motion which the disc is apt to acquire. The general results are, that when the diameter of the disc is something less than twice that of the orifice, it is blown off*, that upon increasing the diameter of the disc larger

* This fact has been also observed by R. Younge, esq. in the *Phil. Mag.* April 1828, p. 282.

weights are required to pull it off, till we reach a certain point beyond which the same or a rather less weight is required, however the diameter of the disc be increased. Also that the weights increase with the increase of the orifice, or of the pressure.

TABLE II.

		Diameter of Disc.													
		.25	.35	.4	.45	.5	.6	.7	.75	.8	1	1.5	2	2.5	3
Diameter of Orifice.	.375							0	13	46	129	247	294	327	348
	.25			0	7	30	55	62		86	119	157	175	188	172
	.125	0	11	15		24	27	28		30	31	31	29	26	25

TABLE III.

Pressure in Inches.	1	2	3	4	5	6
Grains.	45	94	145	203	263	327

To understand these phenomena more perfectly, it is necessary to have some means of estimating the pressure on the lower surface of the disc at any required distance from its center. I shall proceed, therefore, to describe an instrument which is adapted to this purpose.

Fig. 2, a section.

Fig. 3. Plan of the lower surface of *FG* seen from below.

ABCD is the lower plate having a tube *B* opening at its center, and communicating with the wind-chest.

FG a circular piece which performs the office of the disc, and is furnished with three legs of slender wire *rs*, *rs*, *rs*, passing through holes in the lower plate, and by which it is kept in its place with its center coinciding with the axis of the tube, while its distance from the lower plate is determined either by the three screws *t*, *t*, *t*, which rest on the lower plate, or by placing washers of known thickness on the wires *rs*. The diameter of its lower surface is 2.4 inches.

This disc is perforated with a circular aperture *vwx* (Fig. 3.), into which is accurately fitted a plug *H* furnished with a shoulder *pp*, which rests on the upper surface of the disc, and serves to keep the lower surfaces of *FG* and *H* in the same plane, and these lower surfaces being turned flat in a lathe, both from the center of the plug *H*, it is evident that they will accurately coincide in whatever position *H* is turned. On the top of *H* is fixed a water-gage *KLM* which communicates with a small hole *i* (diameter .05) drilled in the lower surface of *H*. The relative positions of *FG*, *H* and *i*, will be better understood by referring to Fig. 3, which is a view of the lower surface of the disc. It will be evident that this hole *i* may be placed at any required distance from the center *FG*, while the gage will measure the difference between the atmospheric pressure and the pressure at that point of the lower surface arising from the blast. An index *z*, traversing a graduated scale, serves to shew the distance of *i* from the center of *FG*.

It appears from this instrument that in general the pressure on the disc will be such as is represented in Fig. 4, where the dark parts represent condensation, the mean tint the pressure of the atmosphere, and the light parts rarefaction.

Proceeding from the center of the disc we come to a circle *Aa*, on every point of whose circumference the pressure equals that of the atmosphere, then to a circle *Bb*, where rarefaction is

at a maximum, then to a second neutral circle *Cc*, and then again to one *Dd*, where the condensation attains a second maximum. Taking these circles as the most characteristic points of the phenomenon, I have, in Tables V. and VI. given their radii together with the amount of the maximum rarefaction and condensation at different distances of the disc from the plate, and with various orifices and pressures. Table IV. shews the pressure indicated by the gage at different distances from the center of the disc, and at different distances of the disc from the plate.

TABLE IV.

Pressure 6 Inches.														
Diameter of Orifice .375														
Distance of Disc from Plate.	Distance of the Gage from the center of the Orifice.													
	.1	.15	.2	.25	.3	.35	.4	.45	.5	.6	.7	.8	.9	1
.5	4.	3.8	3.6	3	2.1	1.2	.55	.25	.13	.1				$\overline{.1}$
.25	4.	3.8	3.35	2.6	1.7	.7	.14		$\overline{.1}$	$\overline{.13}$	$\overline{.2}$	$\overline{.28}$	$\overline{.29}$	$\overline{.3}$
.125	4.1	3.8	3.1	2.06	.7	$\overline{.4}$	$\overline{.8}$	$\overline{1.}$	$\overline{1.08}$	$\overline{1.08}$	$\overline{.9}$	$\overline{.6}$	$\overline{.25}$	$\overline{.15}$
.062	4.6	4.3	3.15	.67	$\overline{2.8}$	$\overline{3.57}$	$\overline{2.95}$	$\overline{2.1}$	$\overline{1.34}$	$\overline{.52}$	$\overline{.2}$	$\overline{.09}$	$\overline{.04}$	$\overline{.02}$
.031	5.	4.75	3.75	$\overline{.7}$	$\overline{5.65}$	$\overline{3.25}$	$\overline{1.53}$	$\overline{.95}$	$\overline{.65}$	$\overline{.3}$	$\overline{.15}$	$\overline{.1}$	$\overline{.06}$	$\overline{.03}$
.024	5.	4.75	4.5	$\overline{.95}$	$\overline{5.2}$	$\overline{2.4}$	$\overline{1.2}$	$\overline{.8}$	$\overline{.53}$	$\overline{.25}$	$\overline{.13}$	$\overline{.05}$	$\overline{.03}$	.03
.018	5.25	5.15	4.55	0	$\overline{4.8}$	$\overline{1.35}$	$\overline{.7}$	$\overline{.5}$	$\overline{.25}$	$\overline{.09}$	.05	.1	.1	.05
.015	5.3	5.25	4.8	.65	$\overline{3.5}$	$\overline{.75}$	$\overline{.33}$	$\overline{.13}$	.01	.15	.2	.19	.18	.17
.012	5.45	5.45	5.05	1.75	$\overline{1.8}$	$\overline{.3}$	.2	.3	.43	.45	.47	.5	.4	.35
.009	5.5	5.5	5.3	2.9	.35	.7	.95	1.05	1.2	1.2	1.1	1.05	.9	.8
.006	5.5	5.5	5.5	4.3	2.65	2.75	2.73	2.65	2.57	2.3	2.2	1.85	1.55	1.25

Note. The pressures less than the atmospheric are marked —.

TABLE V.

Pressure at Orifice 6 Inches.

	Diameter of Orifice .375						Diameter of Orifice .25							Diameter of Orifice .125						
Dist. of Disc from Plate.	Radius of Neut. Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	Radius of 2d Neut. Circle.	2d Maximum Pressure. Rad^s.	2d Maximum Pressure. Am^t.	Pressure over Orifice.	Radius of Neut. Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	Radius of 2d Neut. Circle.	2d Maximum Pressure. Rad^s.	2d Maximum Pressure. Am^t.	Pressure over Orifice.	Radius of Neut. Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	Radius of 2d Neut. Circle.	2d Maximum Pressure. Rad^s.	2d Maximum Pressure. Am^t.
.5	.77						3.2	.72						3.6	.53					
.25	.435						3.3	.45	.9	− .16				4.3	.34	.6	− .01			
.125	.33	.57	− 1.08				3.5	.26	.6	− .5				4.5	.22	.45	− .15			
.062	.26	.34	− 3.6				3.85	.183	.295	− 1.63				4.6	.13	.27	− .52			
.031	.245	.29	− 5.7				4.6	.148	.21	− 4.1	.77		+ .02	4.7	.09	.145	− 1.85	.6		
.024	.248	.28	− 5.6	.95			5	.145	.195	− 4.6	.52		+ .05	4.9	.07	.13	− 2.3	.42		+ .05
.018	.25	.279	− 4.85	.65		+ .1	5.2	.149	.185	− 4.6	.45	.7	+ .15	5.1	.08	.115	− 2.75	.35	.6	+ .12
.015	.253	.278	− 3.8	.485	.8	+ .2	5.3	.15	.18	− 4.	.323	.65	+ .45	5.2	.09	.11	− 2.6	.24	.45	+ .22
.012	.257	.276	− 2.55	.366	.75	+ .5	5.4	.155	.18	− 3.	.26	.6	+ .8	5.3	.09	.1	− 1.35	.15	.28	+ .56
.009	.27	.275	− .2	.28	.52	+ 1.3	5.5	.168	.18	− 1.5	.235	.4	+ 1.55	5.4	.1			.1	.23	+ 1.26
.006		.3	+ 2.65				5.6		.18	+ 1.8				5.5						

Note. The pressures greater than the atmospheric are marked + and those less −.

TABLE VI.

	Diameter of Orifice .375																	
	Distance of Discs.																	
	.5	.25	.125			.062			.031				.01					
Pressure at Orifice in inches.	Radius of Neut. Circle.	Neutral Circle.	Neutral Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	Neutral Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	Neutral Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	2d Neutral Circle.	Neutral Circle.	Minimum Pressure. Rad^s.	Minimum Pressure. Am^t.	2d Neutral Circle.	2d Maximum Pressure. Rad^s.	2d Maximum Pressure. Am^t.
1	.765	.43	.325	.53	$\overline{.18}$	.26	.34	$\overline{.56}$	.253	.3	$\overline{.65}$	.5					.46	.33
2	.77	.435	.327	.55	$\overline{.35}$	.26	.34	$\overline{1.15}$	.249	.3	$\overline{1.57}$	.7	.27	.28	$\overline{.05}$	.295	.46	.43
3	.775	.44	.329	.55	$\overline{.55}$	.33	.34	$\overline{1.7}$	.248	.295	$\overline{2.5}$	.82	.265	.28	$\overline{.4}$	.32	.52	.5
4	.783	.44	.33	.565	$\overline{.7}$	.33	.34	$\overline{2.3}$	.247	.29	$\overline{3.5}$	.84	.262	.276	$\overline{.8}$	.33	.52	.59
5	.79	.437	.33	.567	$\overline{.87}$	.33	.34	$\overline{2.65}$	.246	.29	$\overline{4.6}$	.88	.26	.276	$\overline{1.25}$	.339	.52	.6
6	.77	.435	.33	.57	$\overline{1.08}$	.26	.34	$\overline{3.6}$	.245	.29	$\overline{5.7}$	.89	.26	.276	$\overline{1.8}$	.34	.52	.6

It appears from this Table that upon varying the pressure the radii of the principal circles remain very nearly constant, while the amount of the minimum pressure varies nearly as the pressure applied at the orifice, except when the plates are very near indeed.

The pressure on any circle of the disc is plainly proportional to the pressure indicated by the gage on that circle and to its radius, jointly. The curves in Fig. 6. are constructed on this

principle, their abscissæ representing the distance of the gage from the center, and their ordinates being taken proportional to (pressure) × (distance), the ordinates becoming of course negative when the pressure is less than that of the atmosphere. Hence the area will represent the whole difference between the pressure on the lower surface of the disc and the atmospheric pressure, the upward pressure being proportional to the areas above the axis, and the downward pressure to those below. These diagrams, therefore, shew, by inspection, the variations of pressure at different distances of the disc from the plate.

Thus at .5 it is plainly repelled, at about .25 in equilibrium, from .24 to .024 attracted, between .024 and .018 in equilibrium, and from .018 to contact repelled.

The center of the orifice is in all these figures at the left hand, and the decimals are the distances of the disc from the plate.

To ascertain the effect produced by varying the diameter of the disc, a gage *ABCD*, Fig. 5, was provided, which terminated in a flat horizontal tube, made so thin that it could be introduced between two plates at a distance of .08 from each other, the upper plate being, for the convenience of seeing the point of the gage, of glass; and sustained by three little knobs or feet, which served to maintain its distance and parallelism with the lower plate, and yet allowed it to be moved about into any position with respect to the orifice.

Inferring then that the pressure of the current, estimated at right angles to its direction, would be the same whether measured in a direction parallel or perpendicular to the disc, the end of the gage was placed, as in the figure, at right angles to the radius, and upon moving it to different distances from the center of the orifice, its indications were found actually to agree with those already obtained by means of the gage before described.

Now upon keeping this gage fixed at any distance whatever from the center of the orifice, and observing its indications while the upper plate was moved about, it was found that the pressure was not at all affected by such motion, unless the edge of the upper plate was brought very near the point of the gage, when the pressure became slightly diminished. It may be concluded from this that *cæt. par.* the pressure at any point of the disc at a given distance from the orifice, is not affected by increasing or diminishing the diameter of the disc, or otherwise altering its figure, and this may serve to show why the small discs are blown off: for if in Fig. 4, the disc be reduced to the diameter of *Aa*, there will be no rarefaction, and it must necessarily be repelled. Again, if it be made at all greater than *Cc*, the rarefaction will not be increased, but the condensation will slightly, and therefore it will sustain a rather less weight upon further increasing it.

A good experiment in illustration of all this is one that was devised by Hauksbee, as long ago as 1719. He shewed that when a current of air was made to pass through a small box, the air contained in the box became considerably rarefied. From this and similar experiments it appears that a current of air communicates its motion to the particles in its immediate neighbourhood, and carries them along with it. In our experiment, then, the first portion of air when it issues from the orifice instead of dispersing itself in distinct streams, communicates its motion to the air contained previously between the plates, and carries it away so that the succeeding portions are compelled to fill the whole space. The air may then be considered as issuing in successive concentric annuli from a cylindrical aperture, whose length is the circumference of the orifice, and height the distance between the plates. Now as the particles in each annulus issue with a certain velocity, and in lines radiating

from the center, they must necessarily increase their distance from each other, and hence the air in the annulus becomes suddenly rarefied, by which means its progressive velocity and the pressure on the preceding and succeeding portions is variously modified. Some of these modifications I have attempted to develope experimentally, and have ventured to submit these results to the Society in the hope that they may be found useful hereafter in confirming any theoretical views of the subject which may appear, and in the mean time may serve to throw some light on a phenomenon which is doubtless possessed of very great interest.

R. WILLIS.

Caius College,
April 21, 1828.

Fig. 1.

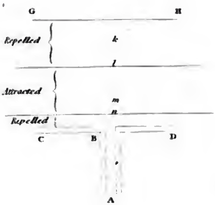


Fig. 5.

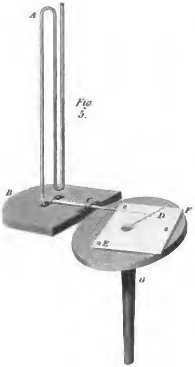


Fig. 2.

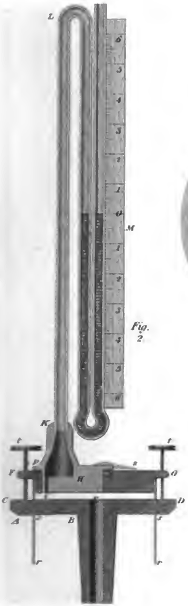


Fig. 4

Fig 3.

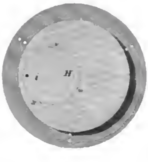
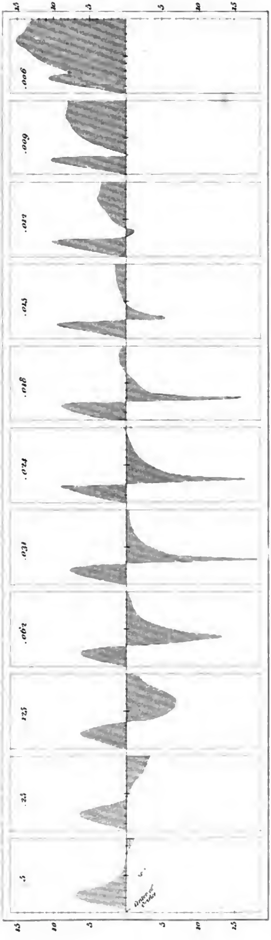


Fig. 6.



J W Lowry sculp

V. *On the Calculation of Annuities, and on some Questions in the Theory of Chances.*

By J. W. LUBBOCK, Esq. B.A.

[Read *May* 26, 1828.]

The object of the following investigation is to shew how the probabilities of an individual living any given number of years are to be deduced from any table of mortality. All writers (with the exception of Laplace) have considered the probability of an individual dying at any age to be the number of deaths at that age recorded in the table, divided by the sum of the deaths recorded at all ages. This would be the case if the observations on which the table is founded were infinite, but the supposition differs the more widely from the truth the less extended are the observations, and cannot, I think, be admitted where the recorded deaths do not altogether exceed a few thousand, as is the case in the tables used in England. The number of deaths on which the Northampton tables are founded is 4689, (*Price*, Vol. I. p. 357.). The tables of Halley are founded upon the deaths which took place at Breslaw in Silesia during five years, and which amounted to 5869.

If a bag contain an infinite number of balls of different colours in unknown proportions, a few trials or drawings will not indicate the proportion in which they exist in the bag, or the simple probability of drawing a ball of any given colour, and not only

the probability of drawing a ball of any given colour calculated from a few observations will be little to be depended on, but it will also differ the more from the ratio of the number of times a ball of the given colour has been drawn, divided by the number of the preceding trials, the fewer the latter have been.

Laplace (*Théor. Anal. des Probabilités,* p. 426.) has investigated the method of determining the value of annuities, he there says, " Si l'on nomme y_0 le nombre des individus de l'age A dans la table de mortalité dont on fait usage et y_x le nombre des individus a l'age $A+x$ la probabilité de payer la rente à la fin de l' année $A+X$ sera $\frac{y_x}{y_0}$," this hypothesis coincides with that I have before alluded to, as adopted by all other writers. Laplace, however, means this as an approximation, for he has investigated differently the probability of an individual of the age A living to the age $A+a$, p. 385 of the same work. He there considers two cases only possible, but as an individual may die at any instant during life, I think it may be doubted whether this hypothesis of possibility should be adopted.

Captain John Graunt was the first, if I am not mistaken, who directed attention to questions connected with the duration of life, he published a book in 1661, entitled *Observations on the Bills of Mortality:* which contains many interesting details although it is written in the quaint style which prevailed in those times. In this book, amongst other tables, there is one shewing in 229250 deaths how each arose: and another shewing of 100 births " how many die within six years, how many the next decad, and so for every decad till 76," which is in fact a table of mortality, and is probably the first ever published.

After Captain Graunt, Sir W. Petty published his *Essays on Political Arithmetick*; Halley, however, was the first who

calculated tables of Annuities, he took the probabilities on which they depend, from a table of mortality founded on the deaths during five years at Breslaw. Since his time a great number of writers have treated of these subjects, of whom a notice may be seen in the Encyclopædia Britannica, or in the Report from the Committee on the Laws respecting Friendly Societies, 1827, p. 94. It is to be regretted that those who have published tables of mortality should generally not only have altered the radix or number of deaths upon which the table is constructed, but also the number of deaths recorded at different ages, in order to render the decrements uniform; this is the case particularly with the Northampton tables, as published by Dr. Price. See Price on *Reversionary Payments*, Vol. I. p. 358. For if observations were continued to a sufficient extent, they would probably shew that some ages are more exposed to disease than others, that is, they would indicate the existence of climacterics, of which alterations such as these destroy all trace.

I annex four tables which I have calculated with the assistance of Mr. Deacon, from the tables of mortality for males and females at Chester, given by Dr. Price, Vol. II. p. 392. The two first tables shew the probability of an individual at any age living any given number of years, as well as the expectation of life at any age. The two last shew the value of £.1 to be received by an individual of any age after any number of years, and the value of an Annuity. The difference between these values for a male and female is very great, and shews that tables which would be applicable for the one would not be for the other.

I have also subjoined a table comparing the values of annuities calculated from observations at Chester (according to the hypothesis of probability I have assumed), with some which have been calculated from observations at other places. Until lately

the Government of this country granted Annuities, the price of which depended on the price of Stock, which renders their tables complicated. I have given their values of a deferred annuity for five years, compared with those I have calculated from the observations at Chester; it will be seen that the former are much too high.

2. Suppose a bag to contain a number of balls of p different colours, and that having drawn

$$m_1 + m_2 + m_3 \ldots\ldots + m_p$$

balls, m_1 have been of the first colour, m_2 of the second colour, m_3 of the third colour, m_p of the p^{th} colour. If x_1, x_2, $x_3 \ldots\ldots x_p$ are the simple probabilities of drawing in one trial, a ball of any given colour, the probability of the observed event, is

$$x_1^{m_1} \times x_2^{m_2} \ldots\ldots x_p^{m_p}$$

multiplied by the coefficient of $x_1^{m_1} x_2^{m_2} \ldots\ldots x_p^{m_p}$ in the development of

$$(x_1 + x_2 \ldots\ldots x_p)^{m_1 + m_2 \ldots\ldots + m_p}.$$

The event being observed, the probability of this system of probabilities is

$$x_1^{m_1} \times x_2^{m_2} \ldots\ldots \times x_p^{m_p}$$

divided by the sum of all possible values of this quantity.

The probability in $n_1 + n_2 \ldots\ldots + n_p$ subsequent trials of having n_1 balls of the first colour, n_2 of the second, n_p of the p^{th}, is a fraction of which the numerator is the sum of all the values of

$$x_1^{m_1 + n_1} \times x_2^{m_2 + n_2} \ldots\ldots x_p^{m_p + n_p},$$

and of which the denominator is the sum of all the values of

$$x_1^{m_1} \times x_2^{m_2} \ldots\ldots x_p^{m_p};$$

multiplied by the coefficient of

$$x_1^{n_1} \times x_2^{n_2} \dots . x_p^{n_p},$$

in the development of

$$(x_1 + x_2 + x_3 \dots . , + x_p)^{n_1 + n_2 \dots . + n_p}.$$

Since $x_1 + x_2 \dots . + x_p = 1$, if x_1, x_2, &c. be all supposed to vary from 0 to 1, and all these values to be equally possible *à priori*, the numerator will be found by integrating the expression

$$x_1^{m_1+n_1} \times x_2^{m_2+n_2} \dots . (1 - x_1 - x_2 - x_3 \dots\dots - x_{p-1})^{m_p+n_p} dx_1 \times dx_2 \dots . dx_{p-1}$$

first from $x_{p-1} = 0$ to $x_{p-1} = 1 - x_1 - x_2 \dots\dots - x_{p-2}$,

then from $x_{p-2} = 0$ to $x_{p-2} = 1 - x_1 \dots\dots - x_{p-3}$,

and so on. The denominator will be found in the same way.

If the coefficient of $x_1^{m_1} \times x_2^{m_2} \dots . x_p^{m_p}$ in the development of

$$(x_1 + x_2 \dots . + x_p)^{n_1 + n_2 \dots . + n_p},$$

be called C, these integrations give for the probability required

$$C \times \frac{(m_1+1)(m_1+2)(m_1+3) \dots . (m_1+n_1)(m_2+1)(m_2+2) \dots . (m_2+n_2) \dots .}{(m_1+m_2+m_3 \dots . + m_p + p)(m_1+m_2+m_3 \dots . + m_p + p + 1) \dots .}$$

$$\frac{(m_p+1)(m_p+2) \dots . (m_p+n_p)}{(m_1+m_2 \dots . + p + n_1 + n_2 + n_3 - 1)},$$

or if the product

$$(m_p + 1)(m_p + 2) \dots . (m_p + n_p)$$

be denoted by $[m_p + 1]^{n_p}$, which is the notation used by Lacroix *Traité du Calcul Differentiél*, Vol. III. p. 121; the probability required is

$$C \times \frac{[m_1+1]^{n_1}[m_2+1]^{n_2} \dots . [m_p+1]^{n_p}}{[m_1+m_2 \dots . m_p+p]^{n_1+n_2+n_3 \dots + n_p}}.$$

This probability is the same as if the simple probability of drawing a ball of the p^{th} colour were $m_p + 1$, with the difference of notation.

When n_2, n_3, n_{p-1}, &c. $= 0$, and $n_p = 1$, this expression gives for the chance of drawing a ball of the p^{th} colour,

$$\frac{m_p + 1}{m_1 + m_2 \dots + m^p + p},$$

and the probability that the index of the colour drawn is between $n - 1$ and $n + q + 1$ is

$$\frac{m_n + m_{n+1} \dots m_{n+q} + q}{m_1 + m_2 \dots + m_p + p}.$$

If we suppose the law of the possibility of life to be such that p cases or ages are possible, *à priori,* m_1, m_n, &c. will be the number of recorded deaths in a table of mortality at those respective ages, and the chance of an individual living beyond the n^{th} age will be

$$\frac{m_n + m_{n+1} \dots m_p + p - n}{m_1 + m_2 \dots + m_p + p},$$

$m_n + m_{n-1} + \&c. + m_p$ is the number given by the table as living at the n^{th} year; therefore, on the hypothesis of this law of possibility, the chance of an individual living beyond the n^{th} year is a fraction of which the numerator is the number living at that age $+ p - n$, and the denominator is the whole population on which the table is founded, or the radix $+ p$. The tables 1 and 2 have been calculated from this formula from observations at Chester, given by Dr. Price, Vol. II. p. 107.; p was taken equal to 101 for a child at birth, that is, the chances of a child living beyond a hundred years, and of its dying in each intermediate year, were supposed to vary from 0 to 1, all these values being equally probable, *à priori*. The value of any sum to be received after any number of years is equal to the sum itself multiplied by the chance of

the individual being alive to receive it, therefore, these tables give the value of unity to be received after any number of years. Considering duration of life to be valuable in proportion to its length, the value of the expectation of life to any individual is the sum of the chances of his living any number of years multiplied by the intervening time, so that if P_n be the chance of an individual living *exactly* n years, the value of his expectation of life, is $\Sigma n P_n$, which is evidently equal to $\Sigma P'_n$, if P'_n be the chance of an individual *surviving* n years; therefore, the value of the expectation of life of any individual is the sum of the numbers on the same line in Tables I. and II. The unity of expectation is here the expectation of an individual who is certain to live exactly one year. The Tables I. and II. give the values of contingencies depending on a single life, without discount; the Tables III. and IV. are the same values, discounted at the rate of 3 per cent. compound interest. These tables give the values of annuities about 6 per cent. higher than those calculated from the Northampton, and given by Dr. Price, Vol. II. p. 54. The only tables that I have met with of annuities on female lives are calculated from observations in Sweden, and are given by Dr. Price, Vol. II. p. 422. But they are calculated at 4 and 5 per cent. interest. It is not to be expected, however, that tables calculated from observations made in one country will serve in another, or even in different parts of the same country.*

The probability of having n_1 balls of the first colour in n_1+N trials, the colours of the other N balls being any whatever, is

$$\frac{\int x_1^{m_1+n_1}(1-x_1)^N x_2^{m_2} x_3^{m_3} \ldots\ldots (1-x_1-x_2 \ldots . x_{p-1})^{m_p} dx_1 dx_2 \ldots . dx_{p-1}}{\int x_1^{m_1} x_2^{m_2} \ldots\ldots (1-x_1-x_2 \ldots\ldots x_{p-1})^{m_p} dx_1 dx_2 \ldots\ldots dx_{p-1}}$$

* Since writing the above, I find that Mr. Finlaison has given the values of Annuities, distinguishing the Sexes, in the Report of the Committee on Friendly Societies, 1825, p. 140.

multiplied by the coefficient of x^{n_1} in the development of $(x_1+y)^{n_1+N}$, the integrals being taken between the same limits as before.

These integrations give for the probability required

$$C\times\frac{(m_1+1)\,(m_1+2)\ldots(m_1+n_1)\,(m_2+m_3+m_4\ldots+p-1)\,(m_2+m_3+m_4\ldots+p)\ldots\ldots}{(m_1+m_2\ldots\ldots+m_p+p)\,(m_1+m_2\ldots\ldots+m_p+p+1)\ldots\ldots}$$

$$\frac{(m_2+m_3+m_4\ldots\ldots+p+N-2)}{(m_1+m_2+m_3\ldots\ldots+p+n_1+N-1)},$$

$$C \text{ being equal to } \frac{(n_1+1)\,(n_1+2)\ldots\ldots(n_1+N)}{1\,.\quad 2\ldots\ldots\ldots\ldots N+1}.$$

Adopting the same notation as before, this probability is equal to

$$C\times\frac{[m_1+1]^{n_1}[m_2+m_3\ldots+m_p+p-1]^N}{[m_1+m_2+m_3\ldots\ldots m_p+p]^{n_1+N}}.$$

$$=\frac{C\,[m_1+1]^{n_1}[m_2+m_3\ldots\ldots+m_p+p-1]^{m_1+1}}{[m_2+m_3\ldots\ldots+N+p-1]^{n_1+m_1+1}},$$

which probability, as before, is the same as if the simple probability of drawing a ball of the p^{th} colour were m_p+1.

If $m_2+m_3\ldots\ldots+m_p+p-2=M$, and if n_1 and N are in the same ratio as m_1 and M, the chance that the number of balls of the first colour in n_1+N trials is between the limits n_1 and $n_1\pm z$, by the reductions given in the *Théorie Anal. des Probabilités*, p. 386, is

$$1-2\sqrt{\frac{(M+m_1)^3}{m_1M(N+n_1)(M+N+m_1+n_1)\,2\pi}}\cdot\int dz\,e^{-\frac{(M+m_1)^3z^2}{2m_1M(N+n_1)(M+N+m_1+n_1)}},$$

e being the number of which the hyperbolic logarithm is unity, and the integral being taken from $z=z$, to $z=$ infinity.

The question of determining the probability that the losses and gains of an Insurance Company on any class of life are contained within certain limits, is precisely similar to this.

It will be seen from the formula

$$\frac{m_n + m_{n+1} \ldots\ldots m_{n+q} + q}{m_1 + m_2 \ldots\ldots\ldots + m_p + p},$$

p. 6. l. 10. that if life were divided into an infinite number of ages or intervals (in which case p is infinite), the hypothesis of possibility remaining the same, the probability of an individual dying in any given interval would be the given interval divided by the whole duration of life, which coincides with that which is given by De Moivre's hypothesis. Thus if life were supposed to extend to a hundred years, the probability of an individual dying in any given year would be $\frac{1}{100}$, and any finite number of observations or recorded deaths would not influence the value of this probability. As diseases, and other causes producing death, are not equally distributed throughout life, the last hypothesis cannot be adopted.

In order to investigate accurately the probability of death at any age, it would be necessary to know the law of possibility. Let $\phi_p x_p$ be the probability of the possibility of x_p, then the probability in the former question of having n_1 balls of the first colour, n_2 of the second, &c. in $n_1 + n_2 \ldots\ldots + n_p$ trials, is

$$C \times \frac{\int x^{m_1 + n_1} (\phi_1 x_1) x_2^{m_2 + n_2} (\phi_2 x_2) \ldots (1 - x_1 - x_3 \ldots x_{p-1})^{m_p + n_p} dx_1 dx_2 - dx_{p-1}}{\int x_1^{m_1} (\phi_1 x_1),\ x^{m_2} (\phi_2 x_2) \ldots\ldots (1 - x_1 - x_2 \ldots x_{p-1})^{m_p} dx_1 dx_2 - dx_{p-1}},$$

ϕ is a sign of function, and this function may be either continuous or discontinuous.

This expression must be integrated between the same limits as before.

The coefficients of the different powers of x_p in $\phi_p x_p$, or the constants in $\phi_p x_p$, will generally be functions of the index p. If the probability of life were known at a great many places,

and if x_{p_1} were the value of x_p at q_1 places, x_{p_2} at q_2 places, &c. the law of possibility might be determined approximately by considering $\phi_p x_p$ as a parabolic curve, of which x_p is the abscissa passing through the points, of which the ordinates are

$$\frac{q_1}{q_1 + q_2 + \&c.}, \quad \frac{q_2}{q_1 + q_2 + \&c.}.$$

3. In the preceding investigations, the results of the preceding trials are supposed to be known; it may be worth while to examine what the probability of any future event is when the results of the preceding trials are uncertain.

Let a bag contain any number of balls of two colours, white and black, suppose m trials have taken place, and let e_n be the probability that a white ball was drawn the n^{th} trial, f_n the probability that a black ball was drawn.

$$e_n + f_n = 1.$$

First let $e_1, e_2 \ldots\ldots e_n$ be all equal, and let x be the probability of drawing a white ball. If a white ball was drawn every time in the m trials which have taken place, the probability in $n_1 + n_n$ future trials of having n_1 white balls, and n_2 black balls, is

$$\frac{(n_1+n_2)(n_1+n_2-1)\ldots(n_1+1)}{1 \,.\, 2 \ldots\ldots\ldots\ldots n_n} \frac{\int x^{m+n_1}(1-x)^{n_2} dx}{\int x^m dx}.$$

But the probability that a white ball was drawn every time is e^m; therefore, the probability of drawing a white ball n times, and a black ball n_2 times on this hypothesis, multiplied by the probability of the hypothesis, is

$$\frac{(n_1+n_2)(n_1+n_2-1)\ldots(n_1+1)}{1 \,.\, 2 \ldots\ldots\ldots\ldots n_2} e^m \frac{\int x^{m+n_1}(1-x)^{n_2} dx}{\int x^m dx},$$

and the probability of drawing n_1 white balls and n_2 black balls will be the sum of the probabilities on every hypothesis, multiplied respectively by the probability of the hypothesis, which is

$$\frac{(n_1+n_2)\,(n_1+n_2-1)\ldots(n_1+1)}{1\,.\,2\ldots\ldots\ldots\ldots\ldots n_2}\left\{e^m\frac{\int x^{m+n_1}(1-x)^{n_2}\,dx}{\int x^m\,dx}\right.$$

$$+me^{m-1}f\frac{\int x^{m+n_1-1}(1-x)^{n_2+1}dx}{\int x^{m-1}(1-x)\,dx}+\frac{m\,.\,m-1}{1\,.\,2}e^{m-2}f^2\frac{\int x^{m+n-2}(1-x)^{n_2+2}dx}{\int x^{m-2}(1-x^2)\,dx}+\&c.$$

This integral being taken from $x=0$ to $x=1$, is

$$\frac{(n_1+n_2)\,(n_1+n_2-1)\ldots(n_1+1)}{1\,.\,2\,\ldots\ldots\ldots\ldots\ldots\,n_2}\left\{\frac{n_2,\,n_2-1,\,n_2-2\ldots\ldots1\,.\,m+1}{m+n_1+1\,.\,m+n_1+2\ldots m+n_1+n_2+1}\,e^m\right.$$

$$+\,me^{m-1}f\,.\,\frac{n_2+1\,.\,n_2\,.\,n_2-1\ldots\ldots2}{m+n_1\,.\,m+n_1+1\ldots\ldots m_1+n_1+n_2}\,m+1\,.\,m\,+\,\&c.$$

$$=\frac{(n_1+n_2)\,(n_1+n_2-1)\ldots\ldots(n_1+1)}{1\,.\,2\ldots\ldots\ldots\ldots\ldots\ldots n_2}\times\frac{1}{m+2\,.\,m+3\ldots m+n_1+n_2+1}$$

$$\{n_2\,.\,n_2-1\,.\,n_2-2\ldots m+n_1\,.\,m+n_1-1\ldots m+1\,e^m+n_2+1\ldots$$

$$2\,.\,m+n_1-1\ldots m+1\,.\,m\,.\,m\,e^{m-1}f+\&c.$$

This series is equal to $\dfrac{d^{n_1+n_2}.\,y^{n_2}x^{n_1}(ex+fy)^m}{dx^{n_1}\,dy^{n_2}}$, when x and y are made equal to 1, and this is equal to $1\,.\,2\,.\,3\ldots\ldots n_1\,.\,1\,.\,2\,.\,3\ldots\ldots n_2\times$ coefficient of $h^{n_1}k^{n_2}$, in the development of

$$(1+h)^{n_1}(1+k)^{n_2}(1+eh+fk)^m$$

$$(1+eh+fk)^m=1+m(eh+fk)+\frac{m\,.\,m-1}{1\,.\,2}(eh+fk)^2$$

$$+\frac{m\,.\,m-1\,.\,m-2}{1\,.\,2\,.\,3}(eh+fk)^3+\&c.$$

$$(1+h)^{n_1}(1+k)^{n_2}=h^{n_1}k^{n_2}+n_1h^{n_1-1}k^{n_2}+\frac{n_1\,.\,n_1-1}{1\,.\,2}h^{n_1-2}k^{n_2}$$

$$+\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}h^{n_1-3}k^{n_2}+\&c.$$

$$+\,n_2h^{n_1}k^{n_2-1}+n\,.\,n_2h^{n_1-1}k^{n_2-1}+\frac{n_2\,.\,n_1\,.\,n_1-1}{1\,.\,1\,.\,2}h^{n_1-2}k^{n_2-1}+\&c.$$

$$+ \frac{n_2 . n_2 - 1}{1 . 2} h^{n_1} k^{n_2 - 2} + \frac{n_1 . n_2 . n_2 - 1}{1 . 2} h^{n_1 - 1} k^{n_2 - 2} + \&c.$$

$$\frac{n_2 . n_2 - 1 . n_2 - 2}{1 . 2 . 3} h^{n_1} k^{n_2 - 3} + \&c.$$

Coefficient of $h^{n_1} k^{n_2} = 1 + m (n_1 e + n_2 f)$

$$+ \frac{m . m - 1}{1 . 2} \left(\frac{n_1 . n_1 - 1}{1 . 2} e^2 + 2 n_1 n_2 e f + \frac{n_2 . n_2 - 1}{1 . 2} f^2 \right) + \&c.$$

The probability required is

$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots n_1 + n_2}{m + 2 . m + 3 \ldots\ldots m + n_1 + n_2 + 1}$$

$$\left\{ 1 + m (n_1 e + n_2 f) + \frac{m . m - 1}{1 . 2} \left\{ \frac{n_1 . n_1 - 1}{1 . 2} e^2 + 2 n_1 n_2 e f + \frac{n_2 . n_2 - 1}{1 . 2} f_2 \right\} + \&c. \right.$$

If there are p different colours, and if m trials have taken place, and $e_{q,p}$ is the chance that a ball of the p^{th} colour was drawn the q^{th} trial, the probability of drawing n_1 balls of the first colour, n_2 of the second, n_p of the p^{th} in $n_1 + n_2 \ldots\ldots + n_p$ future trials, may be found in the same way.

$$\text{Let } e_{1,1} + e_{1,2} + e_{1,3} + \&c. \ldots\ldots e_{1,n} = S_1, e_1,$$

$$e_{1,1}, e_{1,2} + e_{1,3}, e_{1,4} \ldots\ldots \&c. = S_2, e_1$$

(the sum of the products of e_1 two and two together,)

$$e_{1,1}, e_{2,2} + e_{1,2}, e_{2,3} + \&c. = S_1 e_1, S_1 e_2, ^*$$

and so on; then it may be shewn that this probability is equal to

$$\frac{1 . 2 . 3 \ldots n_1 + n_2 + n_3 \ldots\ldots\ldots + n_p}{m + p \ldots\ldots m + n_1 + n_2 \ldots\ldots + n_p + p - 1} (1 + Se_1)^{n_1} (1 + Se_2)^{n_2} \ldots\ldots (1 + Se_p)^{n_p},$$

$1 + (Se_1)$, $1 + (Se_2)$, &c. being expanded by the binomial theorem, and the indices of S written at the foot.

* This is a method of notation which obtains, but it is not meant to imply that $S_1 e_1 S_1 e_2 = S_1, e_1 \times S_1, e_2$.

The method which was used for summing the series in the last page is of very general application, and depends in fact on this principle, that the generating function of the sum of any series is the sum of the generating functions of each of the terms of the series.

If in the last formula, n_2, n_3, &c. $=0$, and if there be only two events possible, and $n_1 = 1$, the probability required is

$$\frac{1 + Se_1}{m+2}.$$

In order to apply this, suppose an individual to have asserted m events to have taken place, of which the simple probabilities are equal and equal to p, and suppose it required to find the probability of his telling the truth in another case, where the simple probability of the event he asserts to have taken place is not known. Let x be the veracity of the individual, the probability of his telling the truth on this hypothesis is

$$\frac{px}{px + (1-x)(1-p)};$$

and the probability of his telling the truth is the sum of the probabilities of his telling the truth on each hypothesis, divided by the number of the hypotheses.

Suppose x to vary from 0 to 1, and all these values of x to be equally probable *à priori*, then the probability of his having told the truth and the event having taken place, is

$$\int \frac{p\,x\,dx}{px + (1-p)(1-x)},$$

taken from $x=0$ to $x=1$, which integral is

$$\frac{p}{2p-1}\left\{1 - \frac{(1-p)}{2p-1}\text{ hyp. log. }\frac{p}{1-p}\right\}.$$

If $p = \frac{9}{10}$, this probability is .81601, generally if $p > \frac{1}{2}$, the assertion that the event has taken place (on this hypothesis of

veracity) rather diminishes the probability that the event has taken place, if $p = \frac{1}{2}$, the assertion does not alter the probability, if $p < \frac{1}{2}$, the assertion rather increases it.

If $p = \frac{9}{10}$, $e = .81601$, let $m = 10$; then $\frac{1 + Se}{m + 2} = \frac{9.1601}{12}$, which is the probability that the individual will tell the truth in another case. If the individual had told ten truths, the chance of his telling the truth in another case would have been $\frac{11}{12}$.

All values of x between 0 and 1 were supposed equally possible; if they are not, let ϕx be the probability of the possibility of any value of x, then the probability of an individual telling the truth will be $\frac{\int \phi x \, d x}{px + (1 - x)(1 - p)}$ divided by $\int \phi x \, d x$, these integrals being taken from $x = 0$ to $x = 1$.

TABLE formed from the Burials in All Saints' Parish, Northampton, from 1735 to 1780. See p. 143. line 12.

	Actual number of Burials	Reduced to radix 11650	
		By the Observations.	As altered by Dr. Price.
Under 2	1529	3798 7/8	4367
Between 2 and 5	362	899 1/4	1034
5 — 10	201	499 3/8	574
10 — 20	189	469 5/8	543
20 — 30	373	926 5/8	747
30 — 40	329	817 3/8	750
40 — 50	365	906 7/8	778
50 — 60	384	954	819
60 — 70	378	939 1/8	806
70 — 80	358	889 3/8	763
80 — 90	199	494 3/8	423
90 — 100	22	54 5/8	46
Total	4689	11650	11650

J. W. LUBBOCK.

Trinity College,
May 26, 1828.

Age.	1	2	90	95	100	Expectation of Life	Calculated by the usual method.	Age.
0	.78353	.69428	.01232	.00345	.00049	29.75345	28.13	0
1	.88609	.81812	.01821	.00314	(99) .00062	36.69540	35.76	1
2	.92329	.87500	.01207	.00284	(98) .00071	40.21306	39.42	2
3	.94769	.91076	.01001	.00230	(97) .00076	42.54615	41.97	3
4	.97159	.94642	.00810	.00162	(96) .00081	43.83928	43.33	4
5	.97410	.95321	.00584	.00083		44.09357	43.20	5
10	.99364	.98819	.00090			42.75204	41.92	10
15	.99250	.98314				38.95786	38.05	15
20	.98817	.97635				35.82561	34.86	20
25	.98529	.97058				32.99367	32.00	25
30	.98757	.97627				30.27118	29.25	30
35	.98555	.97111				27.04332	25.97	35
40	.98174	.96219				24.04172	22.92	40
45	.97684	.95369				21.34876	20.20	45
50	.97208	.94417				18.81444	17.64	50
55	.97142	.94285				16.34857	15.14	55
60	.95535	.90401				13.63392	12.36	60
65	.94970	.90828				12.05917	10.79	65
70	.93680	.85130				9.41263	8.05	70
75	.90303	.82424				8.43636	7.00	75
80	.89108	.78271				6.99009	5.43	80
85	.86538	.75000				5.90384	4.25	85
90	.84000	.68000				4.32000	2.50	90
95	.71428	.57142				2.14285	1.00	95

This87192.

TAears, from 1772 to 1781,

Ang.	Decᵗ.	Age.	Living.	Dec.
2	11	88	21	4
1	11	89	17	3
0	10	90	14	3
0	10	91	11	3
0	10	92	8	3
0	9	93	5	2
1	8	94	3	2
3	7	95	1	1
6	6	96		
0	5			
15	4			

Age.	1	2	95	100	Expectation of Life	Calculated by the usual method.	Age.
0	.83526	.7495	.00580	.00044	34.55535	33.27	0
1	.89738	.8343	.00481	(99) .00053	40.04475	39.54	1
2	.92792	.8832	.00297	(98) .00059	43.65276	43.25	2
3	.95003	.9154	.00192	(97) .00064	45.87700	45.68	3
4	.96358	.9426	.00135	(96) .00067	47.23533	47.11	4
5	.97830	.965[illegible]	.00069		47.99860	47.44	5
10	.99481	.9896			45.69310	45.17	10
15	.99236	.983[illegible]			41.96030	41.36	15
20	.99120	.9824			38.76308	38.10	20
25	.98576	.971[illegible]			35.49329	34.78	25
30	.98737	.974[illegible]			32.79565	32.27	30
35	.98363	.971[illegible]			30.00384	29.26	35
40	.98339	.966[illegible]			27.13292	26.37	40
45	.98190	.963[illegible]			24.29072	23.50	45
50	.98007	.961[illegible]			21.43212	20.62	50
55	.98074	.961[illegible]			18.35900	17.52	55
60	.96782	.928[illegible]			15.09954	14.20	60
65	.96616	.936			12.83834	11.94	65
70	.94144	.871			9.78378	8.81	70
75	.91919	.841			8.12794	7.14	75
80	.88043	.760			6.30434	5.20	80
85	.83116	.714			5.97402	4.81	85
90	.86842	.736			4.55263	3.46	90
95	.69230	.384			2.07692	1.71	95

T's, from 1772 to 1781,

Dec^t.	Age.	Living.	Dec^t.
21	88	35	4
21	89	31	4
21	90	27	4
21	91	23	4
21	92	19	4
21	93	15	4
21	94	11	4
18	95	7	3
12	96	4	3
8	97	1	1
6	98		

Shewing the present Va r Tables.

Age.	1	2	3)5	100	Value of Annuity.	Annuity of which the value is £.100. £.	s.	d.	Value of Annuity calculated from the same Tables by the common method.	Age.
0	.76070	.65442	.5866020	.00002	13.96256	7	3	2		0
1	.86028	.77148	.7095018	(99) .00003	17.35468	5	15	2		1
2	.89639	.82477	.7779017	(98) .00003	19.17322	5	5	3		2
3	.92008	.85847	.8208013	(97) .00003	20.38907	4	18	0		3
4	.95329	.87209	.8475009	(96) .00004	21.15972	4	14	6		4
5	.94572	.89849	.8577005		21.42118	4	13	4	21.283	5
10	.96469	.93146	.8993		21.55443	4	12	9	21.512	10
15	.96359	.92670	.8902		20.38198	4	18	0	20.283	15
20	.95938	.92030	.8817		19.42818	5	2	11	19.285	20
25	.95659	.91486	.8744		18.55566	5	7	9	18.399	25
30	.95880	.92022	.8820		17.67138	5	13	1	17.492	30
35	.95684	.91536	.8743		16.36473	6	2	2	16.134	35
40	.95314	.90695	.8626		15.05567	6	12	10	14.667	40
45	.94838	.87894	.8515		13.81590	7	4	1	13.493	45
50	.94376	.88997	.8384		12.59164	7	18	7	12.316	50
55	.94312	.88872	.8366		11.28375	8	17	2	10.866	55
60	.92752	.85211	.7803		9.63491	10	7	6	9.140	60
65	.92203	.85614	.7987		8.77709	11	7	10	8.220	65
70	.90951	.80243	.7008		6.94786	14	7	10	6.260	70
75	.87672	.77692	.6877		6.17140	16	4	0	5.291	75
80	.86512	.73727	.6251		5.11141	19	11	3		80
85	.84017	.70694	.5983		4.30505	23	4	6		85

This Table shews the v. 11½ *d.*

different Ages.

Age.	
55	719.283
60	598.460
65	484.681
70	379.062
75	282.833

Shewing the present vter Tables.

Age.	1	2	3 95	100	Value of Annuity.	Annuity of which the value is £.100. £.	s.	d.	Value of Annuity calculated from the same Tables by the common method.	Age.
0	.81093	.70652	.6370034	.00002	15.75290	6	6	11		0
1	.87124	.78641	.7250028	(99) .00002	18.42550	5	8	6		1
2	.90089	.83255	.7780017	(98) .00003	20.14850	4	19	0		2
3	.92235	.86288	.8190011	(97) .00003	21.32367	4	13	9		3
4	.93551	.88856	.8500008	(96) .00003	22.11858	4	10	5		4
5	.94980	.90961	.8750004		22.64306	4	8	4	22.624	5
10	.96583	.93281	.900		22.41169	4	9	2	22.439	10
15	.96345	.92747	.891		21.25267	4	14	1	21.235	15
20	.96223	.92602	.890		20.36435	4	18	1	20.323	20
25	.95704	.91575	.876		19.33571	5	3	5	19.265	25
30	.95861	.91879	.880		18.65687	5	7	2	18.583	30
35	.95498	.91537	.875		17.62920	5	13	5	17.534	35
40	.95474	.91127	.86[illegible]		16.56779	6	0	8	16.366	40
45	.95330	.90847	.86[illegible]		15.42043	6	9	0	15.282	45
50	.95152	.90620	.86[illegible]		14.14872	7	1	4	13.983	50
55	.95217	.90628	.86[illegible]		12.58232	7	18	10	12.376	55
60	.93963	.87474	.81[illegible]		10.67266	9	7	4	10.410	60
65	.93801	.88235	.82[illegible]		9.37655	10	13	3	9.080	65
70	.91401	.82158	.73[illegible]		7.26287	13	15	4	6.889	70
75	.89241	.79343	.70[illegible]		6.02689	16	11	9	5.338	75
80	.85478	.71718	.58[illegible]		4.65536	21	9	6		80
85	.80695	.67327	.57[illegible]		4.34792	22	19	7		85

This Table shews the v or 13*s.* 2*d.*

3 per cent. Interest according to

Age.	Dr. Price's Northampton Table.	The Carlisle Table.	Mr. Finlaison's Observations on the Government Annuitants. Male.	Female.	The Chester Tables. Male.	Female.	Age.
0	25.18	38.72			13.96256	15.75290	0
5	40.84	51.25			21.42118	22.64306	5
10	39.78	48.82			21.55443	22.41169	10
15	36.51	45.00			20.38198	21.25267	15
20	33.43	41.46	20.524	22.312	19.42818	20.36435	20
25	30.85	37.86			18.55566	19.33571	25
30	28.27	34.34	19.223	20.720	17.67138	18.65687	30
35	25.68	31.00			16.36473	17.62920	35
40	23.08	27.61	17.082	19.174	15.05567	16.56779	40
45	20.52	24.46			13.81590	15.42043	45
50	17.99	21.11	14.011	16.111	12.59164	14.14872	50
55	15.58	17.58			11.28375	12.58232	55
60	13.21	14.34	10.777	12.560	9.63491	10.67266	60
65	10.88	11.79			8.77709	9.37655	65
70	8.60	9.18	7.425	8.638	6.94786	7.26287	70
75	6.54	7.01			6.17140	6.02689	75
80	4.75	5.51			5.11141	4.65536	80
85	3.37	4.12			4.30505	3.82905	85

An Annuity of £.1, deferred fe, Interest at 3 per cent.
Years, was

Age.	Granted by Government for	Value by Chester Ta Male.
25	12.009	14.17068
30	11.314	13.25503
35	10.542	11.98695
40	9.681	10.73683
45	8.748	9.57270
50	7.935	8.39092
55	6.685	7.09219
60	5.550	5.71220
65	4.336	4.76981
70	3.081	3.39829

l Payments, according to

arlisle ble. d.	The Chester Tables. Male. £.	s.	d.	Female. £.	s.	d.	Age.
	3	15	5	3	1	3	0
	1	10	11	1	6	7	5
	1	10	5	1	7	6	10
7	1	15	4	1	11	8	15
10	1	19	8	1	15	6	20
1	2	5	2	2	0	3	25
1	2	9	0	2	3	9	30
8	2	17	4	2	9	4	35
0	3	6	9	2	16	0	40
4	3	17	2	3	4	1	45
5	4	9	8	3	14	6	50
11	5	6	0	4	10	2	55
9	6	12	1	5	15	4	60
5	7	10	2	6	17	4	65
0	10	0	4	9	9	1	70
	11	13	4	11	16	0	75
	14	17	10	15	17	11	80

ıt. according to the

75	80	Age.
92740	13.94816	0
35407		5
		10
		15
		20
		25
		30
		35
		40
		45
		50
		55
		60
		65
		70
		75

t. according

80	Age.
)1440	0
	5
	10
	15
	20
	25
	30
	35
	40
	45
	50
	55
	60
	65
	70
	75

TABLE VIII.

Shewing the present value of £.1 to be received by any Female for any number of Years, discounted at the rate of 3 per cent. according to the Chester Tables.

Age.	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	Age.
0	3.29369	5.71103	7.72986	9.40552	10.78504	11.90206	12.80065	13.52326	14.09756	14.54980	14.90324	15.17829	15.38031	15.52442	15.61917	15.67027	0
5	4.39286	7.76151	10.82651	13.33337	15.36325	16.99768	18.31084	19.35442	20.17626	20.82252	21.32924	21.69640	21.97709	22.14891	22.24177		5
10	4.50523	8.24456	11.32305	13.81586	15.82299	17.43564	18.71724	19.72652	20.51522	21.12897	21.57981	21.90149	22.11295	22.22701			10
15	4.46403	8.13911	11.11493	13.51099	15.43614	16.96609	18.17091	19.11249	19.84513	20.38340	20.76760	21.02005	21.15619				15
20	4.45784	8.06753	10.97392	13.30880	15.16457	16.62605	17.76814	18.65695	19.30985	19.77568	20.08189	20.24705					20
25	4.38789	7.91000	10.74860	13.00450	14.78101	16.16931	17.24970	18.04337	18.60962	18.98185	19.18261						25
30	4.40964	7.95262	10.76823	12.98550	14.71842	16.06608	17.05756	17.76485	[illegible]	[illegible]							30

TABLE X.

Shewing the Yearly payment to be made by any Male, in order, after any number of years, to be entitled to an Annuity of £.1 for the rest of his Life, according to the Chester Tables.

Age.	5	10	15	20	25	30	35	40	45	50	55	60	55	70	75	80	Age.
0	2.69374	1.39419	.76711	.50030	.33715	.23702	.16119	.12270	.07428	.05281	.03837	.02391	.01367	.00501	.00235	.00096	0
5	3.21489	1.52209	.89246	.57339	.38652	.26384	.18052	.13239	.08136	.05285	.03252	.01901	.01015	.00586	.00300		5
10	3.10089	1.44433	.84031	.53752	.35586	.23904	.15969	.10596	.06864	.04252	.02528	.01412	.00746	.00404			10
15	2.91315	1.35315	.78444	.49197	.31773	.20811	.13519	.08613	.05236	.03030	.01610	.00772	.00374				15
20	2.77219	1.27916	.72707	.44691	.28192	.18576	.11872	.07344	.04432	.02581	.01488	.00435					20
25	2.63143	1.19074	.66155	.39652	.23791	.15039	.08911	.05091	.02691	.01275	.00097						25
30	2.44724	1.08334	.58942	.34484	.20583	.11993	.06755	.03543	.01671	.00191							30
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]								[illegible]

VI. *On the Longitude of the Cambridge Observatory.*

By GEORGE BIDDELL AIRY, M.A.

MEMBER OF THE ASTRONOMICAL SOCIETY, FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, AND PLUMIAN PROFESSOR OF ASTRONOMY AND EXPERIMENTAL PHILOSOPHY, IN THE UNIVERSITY OF CAMBRIDGE.

[Read *Nov.* 24, 1828.]

The methods of determining the difference of longitude of two places may be classed under two heads: those which depend principally upon geodetic measures, and those which are entirely founded on astronomical observations. The circumstances which are favourable to the application of one of these methods are not in general well adapted to the use of the other: and in few instances have both been used for the determination of longitude. But the difference in the facilities of application is not the only, or the most remarkable difference: they are different in principle, are founded on different definitions of terms, and their results, if they should differ, are equally valuable, but are valuable for different purposes.

Before giving a detailed account of some observations made in order to determine by one of these methods the longitude of a station which had previously been determined by the other, it may not be amiss to describe more particularly the nature and capabilities of both methods.

In the geodetic method it is assumed that the earth is a spheroid, of which the axes are known. When the latitude of a point of departure is determined, the latitude and difference of longitude of any other station visible from the first are calculated without great difficulty, from a knowledge of the bearing and distance of that station from the first. The bearing is ascertained, by comparing the direction of the line joining the stations with the meridian as determined astronomically; the distance, by the ordinary geodetic observations. The second station now becomes a point of departure to a third: and by a repetition of this process, the longitude and latitude of places at a great distance from the original point of departure may be ascertained with considerable accuracy.

The advantage of this method consists in the ease with which, in any country that has been accurately surveyed, the longitude of any point can be found by comparing it with two conspicuous objects that have been observed in the survey. The disadvantages are, that errors will increase as the number of intermediate points is increased, that they may accumulate to a sensible amount, and that the method cannot be extended to countries beyond the limits of vision.

The principal astronomical methods are, reciprocal observations of the bearing of two stations; corresponding observations of occultations of stars by the Moon; corresponding observations of eclipses of Jupiter's satellites; corresponding observations of artificial signals; comparison of clocks regulated to sidereal time at the two stations, by transporting chronometers from one to the other; observations of the Moon's distance from the Sun or a star; and corresponding observations of the Moon's right ascension at the time of her transit. The first has been used in the surveys of England and India, rather for the purpose of forming a scale of longitude than for the determination of the

difference of longitude of two important points. The scale of longitude thus formed in England appears to be incorrect: Captain Kater has stated (*Phil. Trans.* 1828.) that the instrument with which the observations were made is not so perfect as had been supposed: and this is the only remark that I have seen which appears in any degree to account for the errors in longitude*. If the observations are correct, there must be some disturbance in the direction of gravity, of that kind which we attribute to local attraction. Occultations give a more accurate determination than any other known method: but the occultations of bright stars occur seldom, and corresponding observations of the occultations of small stars are rarely found. If there are no corresponding observations at both stations, the determination rests on the exactness of the Lunar Tables, and the method is not accurate. The observations of eclipses of Jupiter's satellites, of the lunar distances, and of the lunar transits, far inferior in accuracy, are in other respects subject to the same remarks. Corresponding observations of artificial signals have been used within a few years in determining the difference of longitude of Greenwich and Paris, and in measuring extensive arcs of parallel on the continent: but they are found, though giving pretty accurate results, to require a degree of system and an extent of co-operation which can only be obtained when the observations are made under the immediate authority of the government. And even with this

* One of the most eminent mathematicians of England has attempted to explain them by considering the bearing of a station as determined by the azimuth of the tangent of the geodetic line which joins that station with the place of observation. But it seems plain that the bearing, as determined by the view of one station from the other, must depend on the direction of the ray of light proceeding in a straight line from one station to the other: and that it will therefore be the same as that of the vertical plane belonging to the point of observation, and passing through the station observed. This is the assumption in the calculations of the English survey.

advantage the method has sometimes totally failed. It is, however, that which is on the whole best adapted to the determination of the difference of longitude of two observatories on the same continent or large island. The method of transporting chronometers has been used with great success for finding the difference of longitude of Dover and Falmouth, and that of Falmouth and Madeira: and an expedition is now employed by order of our government in finding the difference of longitude of several stations on the shores of Africa and South America by this method. In practice it is undoubtedly the best of all, when the stations compared are situated so as to admit of an easy sea-voyage from one to the other.

Now there is this important difference between the geodetic method and the astronomical methods to which we have just alluded. It is necessary in the former to assume that the earth is exactly a spheroid of known dimensions, at least that it is exactly a solid of revolution, and the meridian plane of any station is then defined to be the plane passing through the station and through a certain fixed line called the axis of the earth. In the latter, there is no such assumption: the figure of the earth might be of the most irregular kind, yet the determinations would be made with the same ease, and the same independence of the figure, as if it were a perfect spheroid. The method of occultations ought to be excepted from this remark, as the parallax of the Moon which enters into the computation requires an approximate knowledge of the earth's form and magnitude. But all the others are absolutely independent of any knowledge of the earth's form: the definition of meridian which they assume is, the plane passing through the vertical of the station and the celestial pole: and thus the meridian is determined from elements which in no degree refer to any other part of the earth.

If now the earth's figure be not perfectly regular (which the

discrepancies in meridian arcs apply warrant us to assert), it may be expected that the results of one of these methods will not agree with those of the other. The degrees of longitude on the same parallel found by the geodetic method must be equal, because they are assumed to be equal: those found by astronomical methods may be unequal. The scale of longitude determined on a short extent of the parallel by astronomical methods may not apply to the whole. This in fact is found to be the case with regard to the difference of longitude of Dover and Falmouth, as deduced by proportion of distance from that of Beachy Head and Dunnose.

Let us now consider the different purposes to which the results of these different methods ought to be applied. In mapping a country, it is desirable to divide the map by cross lines into equal spaces. This object is of paramount importance: and it is of comparatively small consequence that the longitudes thus assigned should be exactly the same, except for the coasts, as those determined by astronomical operations. But in fixing the longitude of a place in which astronomical observations are to be made that will be compared with those made at some other place, it is only the difference of the apparent time (as determined by transits, equal altitudes, or some equivalent operation) at which the same phenomenon is observed, that at all interests the astronomer. The exact distance in fathoms east or west is of no consequence, provided he knows the number of seconds which elapse between the passage of the same star over the two meridians. For his purposes then the results of the geodetic method are useless, except there is a high probability that they coincide with the results of the methods described under our second head: and if on trying both methods the inferences cannot be made to agree, the geodetic longitude must be rejected, and the astronomical must be adopted.

These observations will explain the reasons which induced me to take the first opportunity of determining astronomically the longitude of the Cambridge Observatory, though it had already been determined geodetically. And the remarks on the inconvenience of some of the astronomical methods, and the inaccuracy of others, will account for the preference of the method used in the present instance. During my residence at the Observatory, with all the care that could be bestowed by a single observer (unassisted indeed, and fully employed with other observations), only one occultation, and one eclipse of Jupiter's second satellite have been observed. The corresponding observation of the former was not made at Greenwich, and any result of the latter would be doubtful to the extent of 10ˢ at least. These methods, therefore, have hitherto been ineffectual. The method of artificial signals, as I have stated, is too troublesome. A series of lunar transits has been observed, but this observation must be repeated long before a good result can be obtained. The only practicable method remaining was that of transporting chronometers; a method which has scarcely been used on land, but which the facility of communication between Cambridge and London seemed to make in this instance equal or superior to all the others.

An opportunity for trying this occurred in October last. Six chronometers had been lent by the government to Professor Whewell, Mr. Sheepshanks, and myself, for the prosecution of some experiments on the intensity of gravity in the deep Cornish mines. The trial of steadiness to which they were subjected was severe, and they sustained it well. These chronometers were still in our possession on my return to the Observatory: and I suggested to Mr. Sheepshanks, who was then residing in London, and of whose co-operation I had no doubt, that before returning them to the Royal Observatory, they might be usefully employed

in determining the longitude of the Cambridge Observatory. Our plan was soon arranged, and immediately put in execution.

The chronometers were carried on the person of a servant, in a belt of particular construction made under the direction of Mr. Sheepshanks, and which had been found remarkably useful in carrying the chronometers to the bottom of a Cornish mine. One of the government chronometers was disabled by the breaking of its mainspring; but its place was supplied by one lent by the maker, Mr. Molyneux, of which we well knew the value. On Oct. 12, about 1 P. M. Mr. Sheepshanks compared each of the chronometers with the transit clock at Greenwich, and brought them to London, where they were received by my servant, who came to Cambridge by the Boston mail. I compared them about 3 A. M. with the Transit Clock at Cambridge, and immediately returned them by the Observatory servant, who went to London by the Times coach. Mr. Sheepshanks received them and compared them a second time with the Greenwich transit clock: they were again sent to Cambridge by the mail, and I again compared them with the transit clock. My servant then carried them again to London, and Mr. Sheepshanks compared them a third time with the Greenwich clock: after which they were deposited at the Royal Observatory.

The chronometers were compared with the clocks by the method of coincidence of beats. As some members of the Society may not have had occasion to use this method, I will endeavour to explain its principles. The transit clock necessarily goes sidereal time very nearly: the chronometers were regulated to mean solar time. And as 365 solar days are equal to 366 sidereal days, the sidereal clock goes faster than a solar chronometer in the ratio of 366 : 365. Consequently, in one second of time the sidereal clock gains on the solar chronometer $\frac{1}{365}$ of a second. If then the clock is behind the chronometer $\frac{1}{91}$ of a second, in

4 seconds it will beat exactly with the chronometer, and in 4 seconds more it will be $\frac{1}{91}$ of a second before the chronometer. During these 8 seconds the beats will strike the ear at so short an interval that no distinction of sounds is perceptible. The business of the observer, therefore, in comparing a chronometer is to note down the time shewn by both at one of these seconds in which no interval of sound can be perceived. He must then wait till the sidereal clock has gained so much on the chronometer that another coincidence of beats can be observed.

Our chronometers beat 5 times in 2 solar seconds, and consequently the coincidence of beats took place at every alternate second for a short time. The next set of coincidences took place when the clock had gained $\frac{1}{5}$ of a second on the chronometer, or at the end of about 73 seconds.

The accuracy of this mode of comparison can scarcely be imagined without trial. I think there is no doubt that a practised ear can determine the instant of coincidence with an error certainly not exceeding 8 or 9 seconds. This implies an error in the comparison not exceeding $\frac{1}{40}$ of a second. When three or more comparisons are made, and the mean taken, it is probable that the error seldom amounts to $\frac{1}{100}$ of a second.

I think it unnecessary to set down at full length every one of the comparisons of the chronometers and clocks, and shall only give the mean of those taken in immediate succession. It is sufficient to state that the number of comparisons has never been smaller than three, and that five have sometimes been used: that these have been carefully examined by differences to discover the errors which almost inevitably occurred in noting them down: that in one or two instances errors have been discovered and corrected: that the means have been compared with the observations which were nearest in point of time, as a check on the accuracy of the means; and that sometimes the same

fraction of a second has been added to, or subtracted from, both times, in order to remove the repeating decimals.

The following are the makers' names and numbers of the chronometers used:

A, Parkinson and Frodsham, 980.
B, Parkinson and Frodsham, 701.
D, Parkinson and Frodsham, 741.
E, Morice, 169.
F, Murray, 516.
G. Molyneux, 801.

Means of comparisons made at Greenwich by [illegible] [illegible]hanks, Oct. 21, about 2[h].

Time by Greenwich Transit Clock.	Reference to Chronometer.	Time by [illegible] [illegible]meter.
h. m. s.		[illegible] s.
16. 5.10	A	[illegible]
16.10.23	F	[illegible].32.38,8
16.16.14	F	2.54. 6,4
16.20.11	F	3. 4. 0,6
16.24.58	[illegible]	2.52.26,8
[illegible]. 7	D	2.50.38,2
[illegible]	G	3.22.10,8

[illegible]nometer and the Greenwich clock to have [illegible]een two comparisons at Greenwich, a simple [illegible]ve the time shewn by the Greenwich clock [illegible]ich that chronometer was compared with the

Means of comparisons made at Cambridge by Professor Airy, Oct. 21, 16^h.

Time by Cambridge Transit Clock.	Reference to Chronometer.	Time by Chronometer.
h. m. s.		*h. m. s.*
5 . 32 . 14	E	3 . 27 . 36
5 . 39 . 14	A	3 . 13 . 6,6
5 . 46 . 23	F	3 . 33 . 14,4
5 . 51 . 40	B	15 . 27 . 40,8
5 . 58 . 43	G	3 . 53 . 47,8
6 . 5 . 19	D	3 . 35 . 13

Means of comparisons made at Greenwich by Mr. Sheepshanks, Oct. 22, 1^h.

Time by Greenwich Transit Clock.	Reference to Chronometer.	Time by Chronometer.
h. m. s.		*h. m. s.*
15 . 27 . 35,75	A	1 . 2 . 5,5
15 . 35 . 8	E	1 . 31 . 9,8
15 . 39 . 7	F	1 . 26 . 40,2
15 . 44 . 47	D	1 . 15 . 17,8
15 . 49 . 47,5	G	1 . 45 . 31,9
15 . 55 . 24	B	1 . 31 . 58,6

Means of comparisons made at Cambridge by Professor Airy, Oct. 22, 15h.

Time by Cambridge Transit Clock.	Reference to Chronometer.	Time by Chronometer.
h. m. s.		*h. m. s.*
5 . 6 . 50	E	2 . 58 . 32,8
5 . 13 . 2	A	2 . 43 . 1,8
5 . 19 . 36	F	3 . 2 . 44,2
5 . 26 . 24	G	3 . 17 . 41,4
5 . 32 . 41	D	2 . 58 . 39
5 . 38 . 44	B	15 . 10 . 47,4

Means of comparisons made at Greenwich by Mr. Sheepshanks, Oct. 23, 3h.

Time by Greenwich Transit Clock.	Reference to Chronometer.	Time by Chronometer.
h. m. s.		*h. m. s.*
17 . 2 . 21	A	2 . 32 . 38,8
17 . 10 . 35,2	F	2 . 54 . 6,4
17 . 11 . 54	E	3 . 4 . 0,6
17 . 20 . 5	B	2 . 52 . 26,8
17 . 24 . 24,5	D	2 . 50 . 38,2
17 . 30 . 35	G	3 . 22 . 10,8

Assuming any chronometer and the Greenwich clock to have gone uniformly between two comparisons at Greenwich, a simple proportion would give the time shewn by the Greenwich clock at the instant at which that chronometer was compared with the

Cambridge clock. Thus corresponding times of the two clocks were found by each chronometer.

Oct. 21, 16^h. Corresponding times of the Greenwich and Cambridge transit clocks.

Reference to Chronometer.	Time by Greenwich Clock.	Time by Cambridge Clock.	Cambridge Clock fast.
	h. m. s.	*h. m. s.*	*m. s.*
E	5.30. 0,77	5.32.14	2.13,23
A	5.36.59,65	5.39.14	2.14,35
F	5.44. 7,71	5.46.23	2.15,29
B	5.49.25,23	5.51.40	2.14,77
G	5.56.28,36	5.58.43	2.14,64
D	6. 3. 4,61	6. 5.19	2.14,39
Mean.	5.46.41,05	5.48.55,5	2.14,45

Oct. 22, 15^h. Corresponding times of the Greenwich and Cambridge transit clocks.

Reference to Chronometer.	Time by Greenwich Clock.	Time by Cambridge Clock.	Cambridge Clock fast.
	h. m. s.	*h. m. s.*	*m. s.*
E	5. 4.34,90	5. 6.50	2.15,10
A	5.10.47,20	5.13. 2	2.14,80
F	5.17.20,31	5.19.36	2.15,69
G	5.24. 8,40	5.26.24	2.15,60
D	5.30.26,07	5.32.41	2.14,93
B	5.36.28,94	5.38.44	2.15,06
Mean.	5.20.37,64	5.22.52,83	2.15,19

The next step was, to determine the error of each of the clocks upon sidereal time at the place to which it belonged. Mr. Pond (to whose kindness on similar occasions I have often been obliged) favored me with an extract from the books of the Royal Observatory, containing the observed transits on Oct. 21, 22, and 23, with the clock errors deduced from them. The following is an abstract:

Oct. 21, by a mean of 12 stars, when the clock was at $21^h.30$, it was $11^s,21$ fast,
22..................2.................................19 . 32.........10,64,
23..................7.................................2 . 54.........10,12.

From these it is easily found that

Oct. 21, when the Greenwich clock was at $5^h.47$, it was $10^s,98$ fast,
Oct. 22 ....5 . 21 ....10,44.

Consequently at these times the Cambridge clock was fast on Greenwich sidereal time by $\left.\begin{matrix} 2^m.\ 25^s,43 \\ 2^m.\ 25^s,63 \end{matrix}\right\}$ respectively.

The Greenwich observations are reduced by means of the corrections in the catalogue contained in the Nautical Almanac. These corrections are not commonly used in the Cambridge Observatory: but in comparative observations of this description it is evidently right to use the same corrections at both places. With these corrections (omitting the observations of Procyon, as its A.R., as given in the Nautical Almanac, is sensibly erroneous), the errors of the Cambridge clock were as follows:

Oct. 21, by Castor, when the clock was at $7^h.26$, it was $2^m.\ 1^s,93$ fast.
..............Pollux .... 7 . 37......... 2 . 1,86
Oct. 22......Arcturus ....14 . 10......... 2 . 1,79,
(same civil day) α Aquilæ.........................19 . 44......... 2 . 2,03,
.... α Cygni.............................20 . 38......... 2 . 2,05,
..............Mean ....13 . 55......... 2 . 1,93.

These observations I consider equivalent to a greater number of observations at the Royal Observatory, as the errors of level, collimation, and deviation from the meridian, were well known, and the necessary correction had been applied, according to the system of the Cambridge Observatory.

The clock's rate as found by Arcturus was +0,36: by α Aquilæ +0,36: by α Cygni +0,39: by the next observation of Arcturus +0,48: the mean is +0,39.

Using this mean rate, and the mean clock error at the mean time just found, we get these errors,

Oct. 21, Cambridge clock at 5.49, the Cambridge clock was fast on Cambridge sidereal time } $2^m. 1^s,80$

Oct. 22.......................... 5.23.. 2 . 2,18.

Comparing these errors of the clock on Cambridge sidereal time with its errors on Greenwich sidereal time as already found, it appears that Cambridge sidereal time is faster than Greenwich sidereal time,

by the first comparison........$23^s,63$,
by the second................23,45.

The mean of these cannot be far from the truth: and I think, therefore, that we may for the present safely state the longitude of the Cambridge Observatory to be $23^s,54$ East of the Royal Observatory at Greenwich.

In the Trigonometrical Survey of England, the steeple of Grantchester church was observed at the two principal stations of Orwell and Madingley, and its longitude was thus determined to be 6′. 9″ East of Greenwich. The meridian mark of the Cambridge Transit Instrument is on that steeple, and consequently the longitude of the Transit Room, as determined geodetically, is 6′. 9″, or $24^s,6$ of time East of Greenwich.

This determination differs from that which we have just found, not less than 1ˢ,06, or 16″ of space. This is an enormous discrepance. An error of 1ˢ could not exist in any observatory: and an error of 16″ in a survey would imply a linear error of nearly 300 yards. Some other cause must be sought to explain this disagreement, and I see but two which can with any probability be brought forward. One is, an error in the process, by which the difference of longitudes has been determined: the other is, an irregularity in the Earth's form, or a sensible local attraction.

With regard to the first of these, I am well convinced that the error of the clock at Greenwich and at Cambridge was known within one-tenth of a second. On taking the mean of several transits, the mere error of observation is insensible: in the quantity which I have mentioned I think that I have made sufficient allowance for the errors in the position of the instrument, &c. The rate of the Cambridge clock was pretty steady: that of the Greenwich clock had altered on Oct. 20, but it appears to have been steady during these observations. The constant difference in the mode of observation of different observers would give a small quantity, which might be added to the former. As to the possible error in the comparison of the clocks, the Society can judge from the details laid before them whether an error can be supposed sufficiently great to reconcile the two determinations. My own opinion is that it cannot exceed two or three tenths of a second. On the whole, it appears to me that by the most violent and improbable combination of possible errors, not more than half the difference can be accounted for.

I am forced therefore to recur to the other cause, and to hold the opinion that the hypothesis of perfect regularity in the Earth's figure is erroneous to an amount far greater than the probable errors of observation.

For the reasons stated in the commencement of this paper, it is clear that the longitude which must be adopted for the purposes of the Observatory, is that which has been determined by the comparison of the clocks. Until some more accurate determination is made, I shall therefore consider the longitude of the Cambridge Observatory as 23ˢ,54 East of the Greenwich Observatory.

The importance of the general question discussed in this paper is perhaps sufficient to warrant me in laying before the Society an attempt, however imperfect, to establish a fact which has some bearing upon that question. And to the members of this Society the determination is not altogether wanting in local interest. But there is another reason which has operated still more strongly in producing this dissertation on the longitude of the Cambridge Observatory. It is the wish and the hope of the present director of that establishment, that it may rise in time to an importance in the Astronomical world, which will make the exact determination of its geographical situation, and of the position of its meridian-plane, not only desirable, but necessary. A determination like that now presented, if it be judged to be accurate, will then acquire a value to which at present it can make no pretensions.

G. B. AIRY.

OBSERVATORY, CAMBRIDGE,
Nov. 20, 1828.

VII. *On the Extension of Bode's Empirical Law of the Distances of the Planets from the Sun, to the Distances of the Satellites from their respective Primaries.*

BY J. CHALLIS, M.A.

FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *Dec.* 8, 1828.]

1. MORE than half a century has elapsed since Bode of Berlin discovered a singular law of the mean distances of the planets from the Sun, according to which if $4 =$ Mercury's distance, $4 + 3$ will $=$ Venus's, $4 + 2.3 =$ Earth's, $4 + 3.2^2 =$ Mars's, &c. No one, I believe, has ever suggested an existing cause of this physical fact; the theory of universal gravitation points to no such law of the distances at which several small bodies will perform revolutions about a much larger; it has in consequence been customary to ascribe the law of Bode to the original arrangement of the planets at the time they were first set in motion. If however it be owing to a constantly operating cause, a similar phenomenon ought always to be observed under similar circumstances:—the satellites, which revolve round their primaries just as the latter revolve round the Sun, ought to obey the same *law of distances.* Should this be found to be the case it would afford some reason to think that the cause of the phenomenon is not incidental but permanent. In endeavouring to ascertain whether the satellites

observe a like law, I have met with success, the more surprising, that, though easily attainable, it had not been anticipated. Before I state the result of my enquiry, I will enunciate the law in more express terms. It is this:—

When several small bodies revolve round a much larger in orbits nearly circular, their mean distances observe with more or less accuracy, the following progression:

$$a,\ a+b,\ a+rb,\ a+r^2b,\ \&c.$$

It is to be observed that the differences between the true distances and those assigned by this progression, are in several instances very considerable in the system of the planets, the only one in which it has hitherto been recognized.

2. The distances of Jupiter's satellites from his centre are proportional to 60485, 96235, 153502, 269983. These distances, diminished by the least, leave remainders 35750, 93017, 209498. The ratio of the second to the first is 2.60, of the third to the second, 2.25. Half their sum = 2.42, which differs from either about one-fourteenth of its own value. Let $a=60485$, $a+b=96235$, and $r=2.42$.

		Empirical Values.	True Values.	Difference.
Then	a =	60485	60485	0
	$a+b$ =	96235	96235	0
	$a+rb$ =	147000	153502	6502
	$a+r^2b$ =	269851	269983	132

The coincidence of the true and empirical values is as near as happens with respect to the planets, and sufficiently exact to warrant the assertion that the law of distances is true for the satellites of Jupiter.

It is well known that the mean motion of the first satellite + twice that of the third = three times that of the second. Now Laplace has shewn, (*Mec. Cel.* Liv. ii. cap. 8.) that if the primitive mean motions of these satellites were near this proportion,

their mutual action must in time have brought about an accurate conformity to it. The law before us would arrange them nearly as they are arranged, and thus cause the mean motions to be nearly such as they actually are. The conclusion of Laplace therefore makes it probable that the deviations from the law of distances are produced, in part at least, by the mutual actions of the satellites, and so far connects the phenomenon with gravitation.

3. Proceeding now to the satellites of Saturn and subtracting from their mean distances the least mean distance, the remainders will be found proportional to 95, 193, 347, 623, 1873, 6101. The ratios of every two taken consecutively are 2.03, 1.80, 1.79, 3, 3.25. These ratios seem to indicate a twofold series: the three first are not much different from each other, but different from the two last, which again do not much differ from each other. The mean of the three values 2.03, 1.80, 1.79, is 1.87; and the mean of 3.00, 3.25 is 3.12. Let $a = 335$, $a+b = 430$, $r = 1.87$, $r' = 3.12$.

		Empirical Values.	True Values.	Differences.
Hence	a	= 335	335	0
	$a+b$	= 430	430	0
	$a+rb$	= 513	528	+15
	$a+r^2b$	= 697	682	−15
$a+b'$,	or $a+r^3b$	= 956	958	+ 2
$a+r'b'$,	or $a+r^3r'b$	= 2272	2208	−64
$a+r'^2b'$,	or $a+r^3r'^2b$	= 6381	6436	+55

It appears by this that the first, second, third, fourth, and fifth are arranged according to one series; the first, fifth, sixth, and seventh according to another. The recurrence of the first and fifth in the two series is remarkable.

If we have rightly inferred in the preceding Article that the derangements of the law of distances are connected with gravitation, may we not ascribe the singular interruption of the law

in this instance to the action of the enormous ring of Saturn? The arrangement which would have taken place but for the ring, appears from the passage of the value of r from 2.03 to 1.79, to be partially disordered, then entirely broken after the fifth satellite: and it is worth observing how the remaining two, which are considerably more distant from Saturn than the others, tend to comply with the law. If our conjecture be admitted, and the anomaly be rightly ascribed to an existing cause, it is natural to suppose that the cause of the law itself is existing.

4. Should the preceding instances be deemed insufficient to establish the law of distances, no doubt I think will be entertained of its reality when the satellites of Uranus have been discussed. Their mean distances are as 1312, 1720, 1984, 2275, 4551, 9101. Subtracting from each of these the least mean distance, the results are 408, 672, 963, 3239, 7789. The ratios of every two taken consecutively are 1.65, 1.43, 3.36, 2.41. Of these the first and second differ by a quantity not greater than has happened in other instances: the remaining two require particular consideration. The cube root of 3.36 is 1.50, and the square root of 2.41 is 1.55. These quantities, being near each other and not very different from the other two ratios, prove that the mean distances diminished by the least mean distance are terms of a geometric series nearly. The mean value of the common ratio is 1.53. The great distance of Uranus, and the apparent smallness of his satellites, which have never been seen but by the most powerful telescopes, leave us at liberty to suspect that there are others besides those already discovered. We know how the law of Bode rendered probable the existence of a planet between Mars and Jupiter. The same law, extending to the satellites of Uranus, authorizes the conjecture that there are two between the fourth and fifth, and one between the fifth and sixth, making in all nine.

Suppose $a = 1312$, $a + b = 1720$, $r = 1.53$.

	Empirical Values.	True Values.	Differences.
Then a	$= 1312$	1312	0
$a + b$	$= 1720$	1720	0
$a + rb$	$= 1936$	1984	+ 48
$a + r^2 b$	$= 2267$	2275	+ 8
$a + r^3 b$	$= 2773$	not observed	
$a + r^4 b$	$= 3547$	not observed	
$a + r^5 b$	$= 4731$	4551	− 180
$a + r^6 b$	$= 6543$	not observed	
$a + r^7 b$	$= 9315$	9101	− 214

It is unnecessary to go through the same process for the planets. I will only observe that the ratios of consecutive distances diminished by the least distance, are in order, 1.823, 1.854, 2.066, 2.051, 1.900, 2.054, and the mean value = 1.958.

5. The foregoing investigation has answered the purpose intended, which was to shew the general coincidence of the arrangement of the satellites with an empirical law. When a law, recommending itself by its simplicity, has been established by observation, and is found to be attended with considerable deviations, we are naturally led to enquire whether the deviations themselves are subject to any law. The following process may in some degree answer this end in regard to the law before us. As any three distances would suffice to determine values of a, b, and r, and different values of these quantities would be obtained from every three that may be fixed upon, I have combined the equations in all the ways they admit of, and taken the mean of the different values. Substituting the mean values in the progression a, $a + b$, $a + rb$, &c. and comparing the distances thus obtained with the true distances, if any of the differences compared with the distances should be found much greater or much less than the others, we shall be directed to some peculiarities

in the corresponding bodies, which their known circumstances and qualities may lead us to find out. The rationale of the process is, that by combining the equations the mean values are affected by the peculiarities of all the distances, and by those most which are most predominant.

Let us first take Jupiter's satellites. Here are four equations; $a = 60485$, $a + b = 96235$, $a + rb = 153502$, $a + r^2 b = 269983$. Consequently four different values of each of the quantities a, b, r, may be obtained by combining the equations three and three.

Equations.	Values of a.	Values of b.	Values of r.
1, 2, 3	60485	35750	2.60
1, 2, 4	60485	35750	2.42
1, 3, 4	60485	41782	2.25
2, 3, 4	40850	55600	2.03
	4)222305	4)168882	4)9.30
	55576	42220	2.325

	Empirical Values.	True Values.	Differences.
Hence a	= 55576	60485	+ 4909
$a + b$	= 97742	96235	− 1507
$a + rb$	= 153683	153502	− 181
$a + r^2 b$	= 283746	269983	− 13763

The differences given by this method are, as was to be expected more considerable than those before obtained. The circumstance most worthy of remark is, that the difference corresponding to the third satellite is much less than the others, some peculiarity is therefore to be looked for in this satellite, and the most obvious is its large size, and consequent superior attraction to that of the others:—it is least disturbed and disturbs most. We are thus presented with another argument tending to shew that the deviations are connected with the gravitation and masses of the bodies.

6. The mean values of r, all obtained by a *like process*, have been found as follows:—for Jupiter 2.42, for Saturn 1.87 and 3.12, for Uranus 1.53, and for the planets 1.96. It will perhaps be remarked that these numbers are near the very simple quantities $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3.

$$1.53 - 1.50 = +.03$$
$$1.87 - 2.00 = -.13$$
$$3.12 - 3.00 = +.12$$
$$2.42 - 2.50 = -.08$$
$$1.96 - 2.00 = -.04$$

The differences are most considerable for Saturn's satellites, the distances of which appear to be most disturbed. The smallness of the differences tempts us to suspect that they indicate deviations from a law which would make the ratios be exactly $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3. Suppose $r=2\frac{1}{2}$ for Jupiter's satellites: we shall then have four equations for determining a and b. Consequently six different values of each of these two quantities may be found by combining the equations two and two.

Equations.	Values of a.	Values of b.
1, 2	60485	35750
1, 3	60485	37206
1, 4	60485	40197
2, 3	58057	38178
2, 4	63141	33094
3, 4	75850	31061
	6)378503	6)215486
Mean	63084	35914

		Empirical Values.	True Values.	Differences.
Hence	a =	63084	60485	− 2599
	$a+b$ =	98998	96235	− 2763
	$a+rb$ =	152869	153502	+ 633
	$a+r^2b$ =	287546	269983	− 17563

The differences compared with the corresponding distances are upon the whole not greater in this case than in the preceding, and the value $2\frac{1}{2}$ is just as probably true as 2.325. The difference corresponding to the third satellite is again the least, and is the more to be observed because the other differences are considerably changed from what they were before. If r would have the value $2\frac{1}{2}$ exactly in a particular case, from which the case of nature is a deviation, and the third satellite, being a maximum cause of the deviation, is itself caused to deviate least, then the result of our calculation is such as should be expected, and affords a probability that the value $2\frac{1}{2}$ has foundation in nature.

7. Let us now treat the satellites of Saturn in a similar manner, $a = 335$, $a + b = 430$, $a + rb = 528$, $a + r^2b = 682$, $a + r^3b = 958$, $a + r^3.r'.b = 2208$, $a + r^3.r'^2.b = 6436$. Suppose $r = 2$, $r' = 3$. From these seven equations twenty-one different values of a and b may be obtained. The mean value of $a = 336$, and that of $b = 82$.

		Empirical Values.	True Values.	Differences.
Hence	a	= 336	335	− 1
	$a + b$	= 418	430	+ 12
	$a + rb$	= 500	528	+ 28
	$a + r^2b$	= 664	682	+ 18
	$a + r^3b$	= 992	958	− 34
	$a + r^3.r'.b$	= 2304	2208	− 96
	$a + r^3.r'^2.b$	= 6240	6436	+ 196

The differences are all considerable compared with that corresponding to the first satellite, which is a very small body. One peculiarity readily presenting itself in regard to this satellite, is its proximity to the ring, the exterior radius of which is 233. The ring may possibly have a deranging effect, similar to what a large body would have, revolving at the distance of the satellite; and at the same time that it may have been the primary cause of the two-fold series of distances, it may act as a disturbing

cause, when once the satellites have fallen into this order, and prevent an exact conformity to it. With respect to the other satellites, the difference compared with the distance is greatest for the third, and next greatest for the sixth. The sixth satellite was first discovered and then the third. It is probable that these are the two largest.

8. Lastly, suppose for the satellites of Uranus $r = \frac{3}{2}$, $a = 1312$, $a + b = 1720$, $a + rb = 1984$, $a + r^2 b = 2275$, $a + r^5 b = 4551$, $a + r^7 b = 9101$.

From these six equations values of a and b may be found in fifteen different ways. The mean value of $a = 1283$, that of $b = 437$.

	Empirical Values.	True Values.	Differences.
Hence a	= 1283	1312	+ 29
$a + b$	= 1720	1720	0
$a + rb$	= 1934	1984	+ 50
$a + r^2 b$	= 2259	2275	+ 16
$a + r^5 b$	= 4601	4551	− 50
$a + r^7 b$	= 8749	9101	+352

The differences are least for the second and fourth satellites, pointing them out as maximum causes of derangement. The second and fourth are the satellites Herschel first discovered, and they are the only ones which have since been seen by other Astronomers. The natural conclusion from this, that they are the largest, renders still more probable the inference drawn from the consideration of Jupiter's satellites, that the large bodies of a system derange the law of distances by their gravitation. Astronomers have doubted of the existence of the other four satellites. This singular law, which the great discoverer of Uranus and of his satellites has aided in establishing, reciprocally confirms the value and correctness of his observations.

9. When we come to treat the distances of the planets in a similar manner, we are presented with some difficulty by the

small bodies, Vesta, Juno, Ceres, Pallas. It is impossible to say what would be the effect of breaking up a single planet into several parts, in the way in which these planets have been imagined to originate from the dismemberment of a single body, how the recomposition of the parts would influence the present distances of the other planets, and at what distance the compound body would revolve. Leaving then these bodies out of consideration, and supposing $r = 2$, there will be seven equations, by which twenty-one different values of a and b may be found.

Let $a = 3870981$, $a + b = 7233323$, $a + rb = 10000000$, $a + r^2 b = 15236935$, $a + r^4 b = 52027911$, $a + r^5 b = 95387705$, $a + r^6 b = 19183050$. The mean value of $a = 4091396$, that of $b = 2912316$.

	Empirical Values.	True Values.	Differences.
Hence a =	4091396	3870981	+ 220415
$a + b$ =	7003712	7233323	− 229611
$a + rb$ =	9916028	10000000	− 83972
$a + r^2 b$ =	15740660	15236935	+ 503725
$a + r^3 b$ =	27389924	27352910 {Mean between the distances of Juno, Ceres, and Pallas.}	+ 37014
$a + r^4 b$ =	50688452	52027911	−1339459
$a + r^5 b$ =	97285508	95387705	+1897803
$a + r^6 b$ =	190479620	191833050	−1353430

Here the difference for the Earth is less than those for the adjoining bodies, and its mass is greater than theirs. But the differences for Jupiter, the largest body of the system, and for Mars, are the most considerable. This circumstance may be referable to the anomaly of the four contiguous small bodies, revolving all nearly at the same distance. It is easy to conceive that such an irregularity may have an effect on the arrangement of the adjoining planets; and this account of the matter is made more probable by the small difference between the distance which the law assigns, and the mean between the distances of Juno, Ceres, and Pallas. These three bodies revolve at distances which

differ very little. Vesta is separated from them by an interval much larger than those by which they are separated from each other. The smallness of the difference may indicate, as in other cases, a cause of derangement from which the distance corresponding to it is exempt.

The four planets Vesta, Juno, Ceres, and Pallas, appear to be an instance of the law for the case in which $r = 1$.

A curious inference, which is equally certain with the reality of the law, may be drawn from it:—There can be no planet nearer the Sun than Mercury, and no satellite nearer the several primaries than the nearest of those in each system, which have been discovered.

10. It will now be easy to express the distances of the satellites by series as simple in form as that which Bode found out for the planets.

For Jupiter's satellites $a=63084$, $b=35914$, $\frac{a}{b} = \frac{63}{36}$ nearly, $= \frac{7}{4}$. Suppose the distance of the second satellite $=7+4=11$. As there is nothing to determine the distance to which a term of the series should be equated, I have selected the second, because I observe that the *minus* differences in Art. 6, are in excess, and it is probable that the terms of the series will more nearly express the distances which would obtain were there no disturbances, if the plus and minus differences more nearly counterbalance each other.

	Empirical Distances.		True Distances.	Differences.
Hence	7	$= 7$	6.91	+ .09
	$7+4$	$= 11$	11.00	.0
	$7+4\times 2\frac{1}{2}$	$= 17$	17.54	− .54
	$7+4\times (2\frac{1}{2})^2$	$= 32$	30.86	+ 1.14

For Saturn's satellites $a = 336$, $b = 82$, $\frac{b}{a} = \frac{82}{336} = \frac{1}{4}$ nearly. Let the distance of the first satellite $= 4$.

Empirical Distances.		True Distances.	Differences.
4	$= 4$	4	$.0$
$4+1$	$= 5$	5.13	$-\ .13$
$4+1\times 2$	$= 6$	6.30	$-\ .3$
$4+1\times 2^2$	$= 8$	8.14	$-\ .14$
$4+1\times 2^3$	$= 12$	11.44	$+\ .56$
$4+1\times 2^3\times 3$	$= 28$	26.36	$+1.64$
$4+1\times 2^3\times 3^2$	$= 76$	76.85	$-\ .85$

For the satellites of Uranus $a=1283$, $b=437$, $\frac{b}{a}=\frac{437}{1283}=\frac{1}{3}$ nearly. Suppose the distance of the *fourth* satellite $=3+1\times(1\frac{1}{2})^2=5\frac{1}{4}$, for a similar reason to that before mentioned in treating of Jupiter's satellites.

Empirical Distances.		True Distances.	Differences.
3	$= 3$	3.03	$-\ .03$
$3+1$	$= 4$	3.97	$+\ .03$
$3+1\times 1\frac{1}{2}$	$= 4\frac{1}{2}$	4.58	$-\ .08$
$3+1\times(1\frac{1}{2})^2$	$= 5\frac{1}{4}$	5.25	$.0$
$3+1\times(1\frac{1}{2})^5$	$= 10\frac{19}{32}$	10.50	$+.091$
$3+1\times(1\frac{1}{2})^7$	$= 20\frac{11}{128}$	21.00	$-.916$

For the planets $a=4091396$, $b=2912316$, $\frac{b}{a}=\frac{3}{4}$ nearly. Suppose the Earth's distance $=4+2.3=10$. Hence

Empirical Distances.		True Distances.	Differences.
4	$= 4$	3.87	$+\ .13$
$4+3$	$= 7$	7.23	$-\ .23$
$4+2.3$	$= 10$	$10.$	$.0$
$4+2^2.3$	$= 16$	15.24	$+\ .76$
$4+2^3.3$	$= 28$	27.39	$+\ .61$
$4+2^4.3$	$= 52$	52.03	$-\ .03$
$4+2^5.3$	$= 100$	95.39	4.61
$4+2^6.3$	$= 196$	191.83	4.17

We have in each instance made use of the value of a and b obtained by the process of combining equations, and in all the

cases the ratio $\frac{b}{a}$ admits of being expressed approximately by very simple numbers. This circumstance is worthy of remark, as it appears not to be accidental.

11. The following conclusions may I think, be fairly derived from the preceding investigations:—

1st. The planets and satellites arrange themselves about their primaries at mean distances, which observe with more or less accuracy the following progression:—

$$a, \; a + b, \; a + rb, \; a + r^2 b, \text{ \&c.};$$

with the exception of the satellites of Saturn which assume a twofold progression of the same kind.

2d. It is probable that the value of r must necessarily be one of the terms of the series 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, &c.

3d. There is some degree of probability that the ratio of b to a may always be expressed by very simple numbers.

4th. It is probable that the deviations from exact conformity to the law are depending on the masses and mutual actions of the revolving bodies, or, as in the case of Saturn, on the action of contiguous bodies.

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TRINITY COLLEGE,
Nov. 10, 1828.

VIII. *On the Focus of a Conic Section.*

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[Read *March* 2, 1829.]

Perhaps no property which is known to pertain to the plane section of a cone, is more striking than that which characterizes the focus. Not being aware of any simple and direct mode in which the existence of so remarkable a point is established, I have felt interested to discover a solid construction which might accomplish this object. The result to which I have been led, is exhibited in the following propositions:

Prop. I.

If a right cone be cut by a plane; and if a sphere be inscribed, touching the plane in a point S, and the conical surface in a circle the plane of which is produced to cut the plane of the conic section in a line RX; the distances SP and PR of any point P in the conic section from the point S and the line RX, shall be to one another in a constant ratio.

Let V be the vertex and VO the axis of the right cone $VBDE$, and let AQP be any plane section of the cone, in which the point P is taken (fig. 1, 2, 3.) Through VO draw the plane VAA' perpendicular to the plane AQP, and let it cut the latter in the line AA', and the conical surface in the lines VA, VB. Describe the circle SEB touching the three straight lines AA', VA, and VB in the points S, E, and B respectively; and, because VO bisects the angle AVB, let the point O in VO be the centre of this circle, and join OB, OS. Then, because the plane VAA' is perpendicular to the plane AQP, and that the straight line OS which lies in the former plane is perpendicular to the common section AA' of the two planes, OS is perpendicular to the plane AQP. Therefore, if a sphere be described from the centre O with the radius OS or OB, it will touch the plane AQP in the point S. Join BE, and let BDE be the circular section passing through BE: join VP, and let it cut the circle BDE in D, and join OD. Then because the triangles VOD, VOB have two sides of the one equal to two sides of the other, each to each, and the included angles OVD, OVB equal to one another, the bases OD, OB are likewise equal, and the angle VDO is equal to the angle VBO, that is, to a right angle. Therefore, if a sphere be described from the centre O with the radius OS or OB, the point D will be in the spherical surface, and every other point of the line VP will be without it, and the same may be shewn with regard to every other slant side of the cone: therefore the sphere so described will touch the surface of the cone in the circle BDE.

Let the lines BE, $A'A$ (produced if necessary) meet one another in the point X, and the planes BDE, AQP in the line RX. Then, because RX is the common section of two planes, each of which is perpendicular to the plane VAA', RX

is perpendicular to AX. From P draw PR perpendicular to RX, and therefore parallel to AA' or AX; and through V draw VF parallel to AA' or PR to meet EB (produced if necessary) in F, and join FD, DR. Then, because FD, and DR are parts of the same common section, (viz. the common section of the plane BDE with the plane of the parallels VF, PR,) FDR is a straight line. Lastly, join SP.

Now, the section of the sphere made by the plane VPS is a circle; and the straight line VP touches this circle in the point D, because it lies in the same plane with it, and meets it in that point only; and for the like reason PS touches it in the point S; therefore PD is equal to PS.

Again, because the triangles PDR, VDF are similar, PD is to PR as VD to VF; but VD is equal to VB, and it has been shewn that PD is equal to PS; therefore PS is to PR as VB to VF, that is, in a constant ratio.

Therefore, &c. Q. E. D.

It need scarcely be added that the point S thus found is the *focus*, and RX the *directrix* of the conic section.

If the plane of the conic section be not parallel to a slant side of the cone, that is, if the section be an ellipse or an hyperbola, a second sphere may be inscribed in the lower segment of the cone (see fig. 1.) or in the vertical cone (see fig. 3.), which shall touch the plane AQP in a second point S', and the conical surface in a second circle, with regard to which the same property obtains. For to this the foregoing demonstration is equally applicable, the letters with dashes being substituted for the others, each for each. Thus we arrive at the other focus and the other directrix, and in these cases it is easy to perceive that AS is equal to $A'S'$, and AX to $A'X'$. The next proposition will shew how readily the inscribing of these two spheres leads to the simple properties by which the ellipse and hyperbola are usually defined.

PROP. II.

If a right cone be cut by a plane which is not parallel to a slant side; and if two spheres be inscribed, as in the last proposition, touching the plane in two points *S* and *S′*, and the conical surface in two circles *BDE* and *B′D′E′*; the sum or the difference of the distances *SP* and *S′P′* of any point *P* in the conic section from the points *S* and *S′*, shall be always the same; the sum in the case of the ellipse, and the difference in that of the hyperbola.

Let *AQP* be the section, and let the spheres be inscribed in the manner already shewn. Then, if *VP* be joined, and cut the circles *BDE* and *B′D′E′* in the points *D* and *D′* respectively, the sections of the spheres made by the planes *VPS* and *VPS′* will be circles, the one touched by the straight lines *PD*, and *PS*, the other by *PD′*, *PS′*. Therefore *PS* is equal to *PD*, and *PS′* to *PD′*, and the sum (fig. 1.) or the difference (fig. 3.) of *PS* and *PS′* is equal to *DD′*, that is, to *BB′* or *AA′*.

Therefore, &c. Q. E. D.

The solution of the following problem, which is easily derived from the construction of Prop. I, will point out a remarkable property by which the focus is still further distinguished from every other point in the plane of the conic section.

PROP. III. PROBLEM.

To find the locus of the vertices of all right cones which have the same given plane section *AQP*.

Take *S* the focus and *A* the principal vertex of the section, and let *V* be the vertex of any right cone of which it is a section (fig. 1, 2, 3.) Join *VA*, and let the plane *VAS* cut the conical surface again in the line *VB*. Then, from the construction

by which the focus was determined in Prop. I, it is evident that the plane VAS must be perpendicular to the plane APQ, and that the circle which touches the three straight lines VA, VB, AS, must touch the last in the point S. Consequently,

1. If the section be an ellipse (fig. 1.) having the axis AA', the difference of VA and VA' will be equal to the difference of EA and BA', that is, to the difference of SA and SA'; and therefore the locus in question will be an hyperbola having the foci A, A', and the principal vertices S, S'.

2. If the section be a parabola (fig. 2.) AV will be equal to the sum of VB and AS, that is, (if SG be taken equal to AS, and if GK be drawn perpendicular to GA to cut VB produced in K,) equal to the sum of VB and BK, or to VK; and therefore the locus in question will be a parabola equal to the given one, having the focus A and the principal vertex S.

3. If the section be an hyperbola (fig. 3.) having the axis AA', the sum of VA and VA' will be equal to the sum of EA and BA', that is, to the sum of SA and SA'; and therefore the locus in question will be an ellipse having the foci A, A', and the principal vertices S, S'.

In every case, therefore, the locus is a conic section, which passes through the focus or foci of the given conic section, in a plane at right angles to the plane of the latter, and has for its focus or foci the principal vertex or vertices of the latter. Q. E. I.

Cor. If two conic sections, having their planes at right angles to one another, pass each of them through the focus or foci of the other, each of them shall be the locus of the vertices of all right cones which have the other for a section.

Thus it appears that the focus is the point in which the plane of the conic section is cut by the locus of the vertices of all right cones having it for a section. In fact, it is not

difficult to perceive, that these vertices, and these only of all points without the plane of the section, have with regard to it properties analogous to those which characterize the focus only of all points in that plane, and which have been demonstrated in Propositions I and II of the present Paper. Thus, in every conic section, the distances PV, PY of any point P from the vertex of the cone, and from the straight line YZ (fig. 1, 2, 3.) in which the plane of the section cuts a plane passing through the vertex of the cone perpendicular to the axis, are to one another in a constant ratio: and, again, in such as have a centre, the sum or the difference of the distances of the vertex of the cone from the extremities of any diameter is always equal to the sum or to the difference of VA and VA', the sum in the case of the ellipse, and the difference in that of the hyperbola.

PIERCE MORTON.

Middle Temple,
Jan. 15, 1829.

Fig. 1.

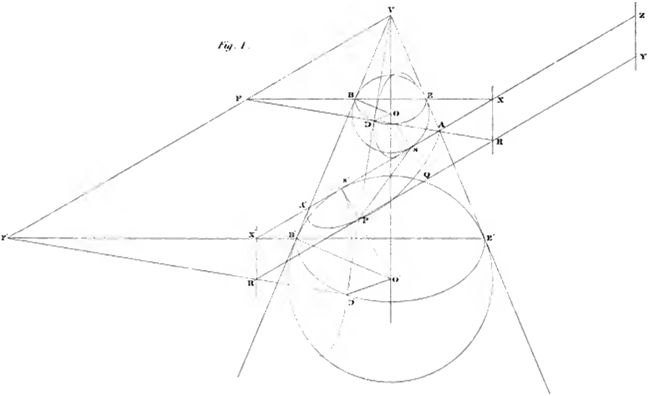


Fig. 2.

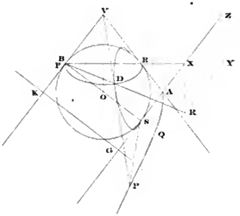
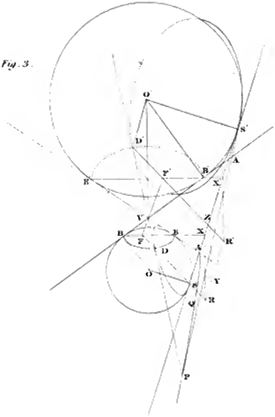


Fig. 3.



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IX. *Mathematical Exposition of some Doctrines of Political Economy.*

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[Read *March* 2 and 14, 1829.]

1. My object in this Paper, is to present in a mathematical form some of the doctrines which have been delivered as part of the science of Political Economy. I am aware that many may at first conceive this to be a frivolous and unprofitable kind of speculation, necessarily barren of any practical and rational results. And this opinion would be undoubtedly true, if it were intended to make mathematical calculations supply the place of moral reasoning; or if it were maintained that we could, by the use of algebraical symbols, obtain any results of a nature different from those which we can obtain otherwise. It is not however with any such views that I now enter upon the subject. But I hope in the course of the following pages to make it appear, that some parts of this science of Political Economy, may be presented in a more systematic and connected form, and I would add, more simply and clearly, by the use of mathematical language than without such help; and that moreover to those accustomed to this language, they may thus be rendered far more intelligible and accessible than they are without it. I hope also to shew, that by this mode of investi-

gation, we may most easily discover what are difficulties of calculation merely, and what, of principle. And in instances where the calculations really are complex and confused, every mathematician is aware how instantly the consideration of them in the general case, points out the course or limits of the simplification which is attainable.

2. It may perhaps appear to some unlikely, that any real difficulty should arise from such causes. But I think that none who have read with attention books on the subject of Political Economy, will be unaware that there is in them often a very considerable complexity of numerical calculation, and no small difficulty in determining how far this is necessary to the argument. I will venture to say also, that some books on these subjects have not escaped fallacies arising mainly or entirely from this complexity, and from the facility of slipping in false principles in the course of such reasonings. It may be allowable to point out one such case.

Mr. Ricardo (*Polit. Econ.* p. 301.) maintains that a tax on wages must fall on labourers; because if it did not so fall, wages would rise, and in consequence of this rise of wages, the price of manufactured goods would rise; and in consequence of this rise of goods, wages would again rise; and so on, without any assignable limits; which he considers to be an absurdity fatal to the doctrine from which it is derived. Now if this argument had been considered mathematically, the absurdity would have disappeared altogether. Let wages rise to the whole amount of the tax; say $\frac{1}{10}$; and let this rise in wages produce a rise of $\frac{1}{20}$ in the price of manufactured goods; (not so much as $\frac{1}{10}$, because only a part, say $\frac{1}{2}$, of the *value* of goods is wages;) and let this

rise of $\frac{1}{20}$ in the price of goods produce a rise of $\frac{1}{40}$ in wages (not so much as $\frac{1}{20}$, because only a part, say $\frac{1}{2}$, of the labourer's *consumption* is manufactures;) and so on. Then it is manifest that the whole rise in wages is $\frac{1}{10} + \frac{1}{40} + \frac{1}{160}$, &c. and that the whole rise in goods is $\frac{1}{20} + \frac{1}{80} + \frac{1}{320}$, &c. And the simplest principles of systematic arithmetic informs us, that these are not quantities having "no assignable limit," but that according to these suppositions, the whole rise of wages will be $\frac{4}{39}$, and the whole rise of manufactured goods $\frac{2}{39}$. And if the result of these suppositions had been determined by expressing the conditions algebraically, we should have had the same result directly and immediately, without even the process which we have had to introduce, of summing a geometrical series*.

3. It is clear that the proper remedy for such mistakes into which even acute and ingenious men may be led, not knowing or not using mathematical rules, is to make all such calculations the business of a systematic process, which from its nature, cannot neglect any proper numerical considerations, or leave the accuracy of the result questionable. Such a system of calculation must of course borrow the elements and axioms which are its materials, from that higher department of the science of Political Economy, which is concerned with the moral and social principles of men's actions and relations. These materials thus received, stated in the simplest manner, must be subjected to the processes of a proper calculus, and we may thus obtain all the results to which

* This calculation will be given at the end of the paper.

the assumed principles lead, whatever be the complexity of their combination. And such a mode of proceeding will be of very great advantage to truth; inasmuch as it will make it inevitably necessary to separate the moral axioms and assumptions on which the theories rest, from all other matter which may tend to obscure or confound them. It will also separate entirely the two parts of the subject which it is of immense importance to keep separated;—the business of proving these assumptions, and that of deducing their conclusions. Much ingenuity has been shewn in reasoning downwards from assumed principles. These principles are however so few and general, (we do not now speak of their truth or applicability) that the task of deducing their results is almost entirely a business of the mathematical faculty; and might have been done in a few pages by clothing them in mathematical formulæ.

4. It would seem indeed as if the present state of the science of Political Economy were that which peculiarly called for this rigorous and scientific form to be given to its mathematical portions. It is by some reduced to a set of principles hardly more numerous or less general than the laws of motion: and the cases to which the economical principles are applied, are certainly not less complicated than the cases to which mechanical principles are applicable. Now we can easily imagine what would have been the result if men had, without the aid of consistent mathematical calculation, attempted to make a system of mechanical philosophy. There would have been three errors difficult to avoid. They might have assumed their principles wrongly; they might have reasoned falsely from them in consequence of the complexity of the problem; or they might have neglected the disturbing causes which interfered with the effect of the principal forces. And the making mechanics into a Mathematical science supplied a remedy for all these defects. It made it necessary to state

distinctly the assumptions, and these thus were open to a thorough examination; it made the reasonings almost infallible; and it gave results which could be compared with practice so as to shew whether the problem was approximately solved or not. It appears I think that the sciences of Mechanics and Political Economy are so far analogous, that something of the same advantage may be looked for from the application of mathematics in the case of Political Economy. And this must be remarked, that in this we are so far from claiming for it the rank of a science mathematically demonstrated, that we do not thus assert it to be a near approximation to the business of the world, any more than the doctrines of Mechanics are to actual practice, if we neglect friction and resistance, and the imperfection of materials, and suppose moreover the laws of motion to be questionable. But we hold this method of investigation to be the best way of separating the theories which have been advanced, into the different kinds of truth, or of falsehood, of which they may happen to consist. And we conceive, that by doing this we shall better enable men to form their opinions of the value of these several parts.

5. The portion of the subject on which I wish at present to treat, belongs to the second of the processes liable to error, of which I have mentioned three; namely, the deduction of conclusions from fundamental propositions. There are certain reasonings concerning rent and taxes, which are urged by their respective advocates; each founded on the same very simple principles; and each leading to various conclusions: the conclusions of one set of reasoners are opposite to those of another. The principles of the different parties are so nearly identical, that it is manifest that their difference arises from some peculiarity introduced in a subsequent part of the investigation, and it is my intention to employ the processes of mathematics to point out where this latent assumption may reside.

The main point in question is the manner in which wages, profits and rents are affected by certain changes, and particularly by certain taxes. One party, for instance, the followers of Mr. Ricardo, &c. maintain that all taxes on the produce of land are ultimately paid by the consumers: others, and especially Mr. Thompson of Queen's College in this University, hold it demonstrable that they fall principally upon the landlord. It will easily be seen, that it is not now my business to adduce any new arguments on one side or on the other: but to express in mathematical language, so far as they are capable of it, those which each party has urged, and to point out where the divarications occur by which they deviate so widely from each other.

6. It is far from my present intention to write a mathematical theory of Political Economy, or even of any complete branch of it; but it will be necessary to begin with one or two axioms or definitions; in whichever light they may be considered. And first with regard to Rent, I shall premise the following principles:

AXIOM 1. *Rent is the excess of the produce of the land above the usual profits of the capital which is employed upon it.*

This is now the universally acknowledged measure of rent. On the first promulgation of the doctrine, it was stated, that the measure of rent was the excess of the produce of the soil above the produce of the *lowest soil* permanently cultivated; but this enunciation of the principle has since been exchanged for the more correct one, which has just been given. The excess above the produce of the worst soil is a true measure when it coincides with this one, as on the suppositions made by its proposers it does, and it is true no farther.

That the rent is the excess of the produce above the usual profits of the capital is proved by this consideration. The tenant being considered as a person who employs capital and lives on the profits, his profits cannot be different from those of other capitalists.

If they were greater, more capital would be transferred from less profitable employments to farming, the produce would be increased, the price of produce diminished, and thus farming profits would decrease till they were on a level with the profits of other employments or below them. If again farming profits were below other profits, capital would be transferred from agriculture to other employments, where more could be made by it, agricultural produce would diminish, the price to the remaining agriculturists would rise, and with it their profits, till the equilibrium was restored. Thus the farmer would be content and compelled to follow his employment for the same profits as other capitalists. He could not, permanently, get more; he would not, permanently, take less. He would therefore be willing to take land on any conditions which would leave him such profits: he would be willing to give the surplus to the landlord. And whenever the bargain came to be made between them, its true and balanced condition, to which all arrangements would tend, would be, that the landlord should receive this surplus and no more; that is, this surplus would be rent.

It will be perceived, that in this view of the matter the cultivator is assumed to be a person who is remunerated by the profits of his capital for the employment of it. Capital is considered to be transferable from one employment to another whenever the other is more profitable. The conclusions of our reasonings will be true only so far as these assumptions are true.

7. The next principle which I shall enunciate is this:

Axiom 2. *When the produce of any land would equal or exceed the usual profits of the capital requisite to cultivate it, it will be cultivated; and not otherwise; and when new capital can be applied to land so that the additional produce will equal or exceed the usual profits of the capital, the capital will be so applied; and not otherwise.*

This depends on the suppositions already mentioned, that there are capitalists ready and able to employ capital whenever the usual profit can be made of it.

The application of fresh capital to land already cultivated, is in its effects precisely the same as the extension of cultivation to uncultivated lands. Thus if by an annual expenditure of 100, the farmer can obtain a produce of 150 on a given piece of ground, and if profits be 10 per cent., the surplus produce being 50, 10 will go to the farmer and 40 to the landlord as rent. If now a mode became known, by which an *additional* capital of 100 employed on the same ground will give an *additional* produce of 115, this capital will be so applied, and of the resulting produce the farmer will take 10 and the landlord 5: and the result will be the same to all parties, as if *another* piece of ground had been cultivated with a capital of 100 and a produce of 115. We may consider the acre as consisting of two acres, one *superimposed*, on the other; one cultivated with a capital of 100 and produce of 150, the other with a capital of 100 and produce of 115. These successive applications of capital to the same land, are called by Mill, *doses* of capital; and all that can be said of soils of different fertility, may be said also of *doses* of agriculturally employed capital which give different returns.

This however is not a necessary way of stating the subject: for we may with equal propriety consider the case in which the additional capital is employed, as a case of capital 200, produce 265, profit 20, rent 45. The introduction of the gradation of *doses* is an artifice which is unnecessary, if we consider the whole capital and the whole produce on each part of the soil.

8. There is a third principle which has sometimes been introduced into the reasonings on this subject, which seems not to be necessarily or universally true, but which will be assumed in some of the succeeding speculations, because it has been

made one of the foundations of the doctrines which have been maintained.

Axiom 3. *If from any cause the value of the produce of a given quantity of land increases, new land will be cultivated, or new capital will be employed on land already cultivated: and this will be done till there is some land cultivated, or some capital agriculturally employed, which returns no more than the usual profits of capital without any surplus.*

A soil which is in the condition described in the last clause pays no rent, and being in its quality the limit of the soils which can be cultivated, it may be called the *limiting soil;* including in this term also the *last* portion or *dose of capital* employed on land previously cultivated; viz. that portion which produces only common profits.

It is convenient for our calculations to assume that there is always such a limiting soil and limiting dose of capital. It is however not self-evident that this supposition is generally exact.

The supposition would be true, if there existed an indefinite number of kinds of land, differing by insensible shades of productiveness; or an indefinite number of ways of employing successive additions of capital, gradually diminishing in productiveness. The assumption of such a series of soils seems to be somewhat arbitrary: and the supposition of such a list of known ways of employing additional capital on the same soil, seems to be not at all obvious in theory, nor confirmed by the history of the progress of agriculture.

9. It is also necessary to premise some principles on the subject of *prices,* as influenced by *demand* and *supply. Supply* is of itself a quantity, and offers us no difficulty in measuring it theoretically: but *demand* is of a more untangible and fugitive nature. It consists originally of moral elements as well as physical:

of the vehemence of desire, and the urgency of need which men have, as well as of their extent of means. A member of this University who formerly applied mathematical reasoning to some questions of this kind, proposed to measure demand by considering the latter element (the means of purchasing) as constant, the desire and the difficulty balancing each other so as to preserve this constancy.

He supposed (merely as a means of reducing questions to calculation) that men set aside *a certain sum* to purchase a given article, and that this sum measures the *demand*. If the article increase in price they buy just so much less, if it become cheaper they purchase so much more. The price therefore will vary *inversely as* the supply, (not speaking with popular laxity, but in mathematical strictness.) Now this mode of beginning the reasoning, answers extremely well the purpose of avoiding all indefiniteness, and offers a kind of approximation to the law which really obtains as to changes of prices; but probably the approximation is a very loose and inaccurate one. According to this estimate, the failure of $\frac{1}{4}$ the crop of corn, for instance, would increase the price by $\frac{1}{3}$. It is nearly certain that the increase would be much greater. A statement has been made by some writers which we may use as affording a nearer approximation than the one just mentioned, and as shewing the nature of the dependence of the quantities concerned, without resting any thing upon its exactness. According to the writers now referred to, we have the following progression*:

A defect in the harvest of $\frac{1}{10}$ raises the price of corn $\frac{3}{10}$.
.... $\frac{2}{10}$ .... $\frac{8}{10}$.
.... $\frac{3}{10}$ .... $\frac{16}{10}$.

* See Tooke, on High and Low Prices, p. 285.

A defect in the harvest of $\frac{4}{10}$ raises the price of corn $\frac{28}{10}$.

.... $\frac{5}{10}$ .... $\frac{45}{10}$.

There can be little doubt that this table is at least so far true, that the increase of the price is much more rapid than the diminution of the supply. And according to this table the increase of the price varies according to no simple power of the defect of the supply. It would be easy to find the law by which the increase of price may depend on the defect and the square of the defect of supply, so as nearly to satisfy the above data; but it is not necessary for our purpose to go into this detail. In most cases which we shall consider, the diminution of supply and the increase of price, or the contrary events, will be small; and in these instances we cannot be far wrong in assuming the following Axiom.

AXIOM 4. *The increase of price is proportional to the deficiency of the supply.*

Accordingly the fraction which expresses the deficiency must be multiplied by some constant number, in order to give us the fraction which expresses the increase of price. In the table just mentioned, $\frac{1}{10}$ defect of supply gives $\frac{3}{10}$ increase of price. It may therefore perhaps be assumed, that up to the magnitude of $\frac{1}{10}$, the increase of price is three times the deficiency of supply. Thus a deficiency of $\frac{1}{60}$ would increase the prices $\frac{1}{20}$, and so on. I shall however use the general number e instead of 3 in most cases.

It is manifest, that whatever more accurate data we may hereafter obtain for establishing the law of this dependence, the increase of price *ceteris paribus* must be a *function* of the defect of supply, and may be in this manner introduced into the calculation.

10. But there is also another view of the law of price, which it is necessary to consider. No article will be permanently pro-

duced except it pays the cost of production with the profit; if it pay less, the capitalist will abstract his capital from such an employment, the supply will be diminished, price will rise. No article again which any capitalist can produce will permanently have a price higher than that which repays the cost of production with profits. If it have, capital flows to that employment, supply increases, prices fall. Hence it appears, that the cost of production determines permanent price, and this is our fifth Axiom.

AXIOM 5. *The rate of price of any article will be such that the whole price of any portion is equal to the capital employed to produce it together with the usual profits.*

We here speak of the permanent profits, and indeed in the whole of these speculations we consider, as authors on the subject are in the habit of doing, a theoretical condition in which these Axioms are exactly satisfied. This is a state of equilibrium to which in practice things are perpetually tending, but which they never reach.

11. Having thus two Axioms on the subject of prices, it may be desirable to see how they are consistent. Price is determined by the conflict of supply and demand; price is also determined by the cost of production, in which latter expression demand is not mentioned: how then do these agree? In answer to this it is to be observed, that the former is the immediate, the latter the permanent, determination of price. The price to-day is that which arises from the bargaining of to-day's buyers and sellers, that is, from the intensity of demand and the extent of the supply. But this price cannot be long above or below the cost of production, for the reasons already mentioned. This cost is the permanent and ultimate regulator of price. And the demand will affect the *extent* of supply; and if the cost of production vary with the extent of production, as, for instance, in the case of land of different fertilities, the demand affects the cost, and the

two determinations ultimately run together. The point at which they will meet depends on the law of each; and the proper elements being known, may be determined by an equation, as we shall see.

12. It may be proper to add another Axiom.

Axiom 6. *If any tax be imposed on one employment of capital, (for instance, on agriculture), the profits of the capitalist will not be affected by it.*

The word *tax* is here employed for the sake of conciseness and distinctness for any charge or claim, by whatever rule governed, which does not belong to the usual distribution of the produce into rent, profits and wages. It may, according to the quarter where it is found to fall, be considered as part of rent or profits, or of the price paid by the consumer; and according to these circumstances and to the ground of the claim, it may, in many instances, be improperly designated according to common apprehension, by the word which I have adopted as most convenient and precise.

On the supposition already made, that the capitalist can and will withdraw his capital when his profits are diminished, the above Axiom is manifestly true. And the surplus of the produce after the taxes are paid out of it, will therefore be divided in the same manner as the whole was when there was no tax.

This sixth Axiom however is true no longer than while it is allowable to make the supposition above-mentioned. If from any cause the possibility of obtaining the usual profits of capital is affected, the rate of profits will change. If when the profits of the agricultural capitalist are diminished, and he withdraws his capital to transfer it to the field of manufactures, it happen that the quantity of capital which must be transferred to restore the equilibrium, bears a proportion not inconsiderable to the whole capital already employed in the new department, it will no longer be possible to obtain the same profits as before, both

for the capital already present, and for the influx. And in this case the effect of competition will be, that some persons will be found willing to employ capital for a less profit than that which was till then the usual rate, and the rate of profits will fall. If from the excess of the power to produce over the disposition to consume, or from any other cause of the accumulation of wealth, capital increases faster than the means of employing it at the old rate of profits, that rate will in the long run fall. This may be considered as an increase of the *supply* of capital, without a proportional increase of the *demand*. And the profit, which is the *price* of the employment of capital, will fall with the increased supply, according to laws resembling those which regulate the price of exchangeable articles. According to what rule the rate of profits will fall, with the accumulation of capital, we have no means of determining with any accuracy; but the dependence of the two quantities is manifest. Men will cease to employ capital profitably when they prefer spending it in the manner in which the state of luxury allows, to increasing it in the manner in which the state of trade allows; and this is the cause which checks the fall of profits by checking the accumulation on which it depends.

I shall not at present consider changes in the rate of profits. This rate will be assumed to be a constant element. But in reference to the further prosecution of this application of mathematics, it is evident, that the same principles which we shall now have to employ in the case of supply and price, will point out also the path which we must follow, when we undertake to trace the effects of accumulating wealth and declining profits, and the consequences which result with respect to rent, &c. from combining this change with those which are at present the subject of our consideration.

It may be observed also, that the place which wages occupy in this theory, is that of a mode of employing capital; and their variations will depend on the supply of capital directed to such an employment, along with the other causes which influence their amount, and which do not belong to our present investigations.

13. It follows from the 6th Axiom, that in determining the price as in Art. 10, by the cost of production, we must consider, as forming part of the cost, the tax paid. The price must be such as to replace the capital with the usual profits and the tax; for otherwise the capital would desert this employment.

14. We have therefore six Axioms, viz.

(1) Rent is = produce − profits.
(2) Land will be cultivated if produce = or > than profits.
(3) There is always a *limiting soil*.
(4) Increase of price $\propto$ diminution of supply.
(5) Price = cost of production + profits.
(6) Taxes do not affect the rate of profits.

The results of the combination of these six principles, may, as has been said, be most certainly and easily deduced by means of mathematical reasoning, and we proceed to trace the consequences of them in the case of taxes imposed on the produce of land, as tithes, land-tax, rates, &c.

15. Let it be supposed that there are various qualities of soil, which we may call the 1st, 2nd, 3rd ... m^{th} ... n^{th}: that the quantities of each of these soils are respectively a_1, a_2, $a_3 \ldots a_m \ldots a_n$ acres or units of land: that the capital employed on one acre in the different cases is respectively c_1, c_2, $c_3 \ldots c_m \ldots c_n$ shillings, or units of money: that the produce of one acre of each quality, is respectively r_1, r_2, $r_3 \ldots r_m \ldots r_n$ quarters or units of produce. Let it be supposed also, that the price of a quarter (or other unit) of

corn is p shillings (units of money), and that the annual return requisite to replace a capital c with the usual profit, is qc ($q-1$ being a fraction which expresses the rate of profit.)

In general, we shall have to consider only the *average* produce and rate of all the soils, except the last quality; the last quality being that which is brought into cultivation, or thrown out, by the changes which we have to consider.

Let a be the whole number of acres in cultivation, c the average capital employed on an acre, r the average produce per acre. Then the whole produce is ar, and its price arp; the whole capital is ac, and the sum requisite to replace it with profit is acq; and hence by the 1st Axiom, the whole rent is $arp - acq$, it being supposed that there are no taxes.

If we now suppose taxes (viz. a tax affecting agricultural undertakings only) to be paid on each acre, depending in any manner on the quality; viz. on the 1st quality t_1 per acre, on the second t_2, on the m^{th} t_m, on the n^{th} t_n: the whole tax will be $a_1t_1 + a_2t_2 + a_3t_3 \ldots + a_mt_m + \ldots a_nt_n$; and we may suppose this to be at, where t is the average tax per acre: (t_1, &c. are expressed in units of money.)

The rent will now be the excess of the price of the produce above the deductions, which are the tax and the profits on the capital; for the capitalist cannot pay more or less, as will appear by the same reasonings as those which were used when there was no tax. That is, the whole rent will now be

$$arp - at - acq.$$

In this case a, r, p, c, may be different from what they were before, in consequence of the introduction of t, and we have to examine what the alterations are which will thus take place: q is supposed to be unaltered by Axiom 6.

In consequence of the tax, let it be supposed that the price p becomes p'; and that the last quality of soil a_n is thrown out of

cultivation. Hence the whole number of acres in cultivation will now be $a - a_n$, and the whole produce $ar - a_n r_n$. Let the produce of the last quality be of the whole produce a fraction u, so that $ar - a_n r_n = ar(1 - u)$, the produce during the tax. Also the capital ac will be diminished by $a_n c_n$, which we will suppose to be a fraction v of the whole capital, and hence the capital employed during the tax will be $ac(1 - v)$. Hence we have

	Without the Tax.	With the Tax.
Rent	$arp - acq$	$ar(1-u)p' - (a - a_n)t - ac(1-v)q$.
Return and profits	acq	$ac(1-v)q$.
Tax	0	$(a - a_n)t$.
Whole price of the produce	arp	$ar(1-u)p'$.

The rent and return with profits in the first case, and the rent, return with profits and tax in the second, are equal to the whole price of the produce.

Hence we have by the imposition of the tax

Diminution of rent $= arp - ar(1-u)p' + (a - a_n)t - acqv$.

If the whole tax be of the whole price of the produce a fraction k, constant or variable with the variations of price, &c.

Diminution of rent......... $= arp - ar(1-u)(1-k)p' - acqv$.
Diminution of profits, &c. $=$ .... $acqv$.
Increase of price.......... $= ar(1-u)p' - arp$.
Tax.......................... $= kar(1-u)p'$.

The tax therefore is the sum of the diminution of rent, the diminution of return to capital, and the increase of price; that is, of what is taken from the landlord, what is taken from the capitalist, and what is taken from the consumer, as it manifestly must be: and we have to consider how these portions are determined.

The diminution of the capitalist's share arises from the diminution of the capital employed, by the throwing of the land

a_n out of cultivation; the rate of profits q is supposed to remain the same as at first: if this vary however, the consequences of its variation may be traced afterwards.

Various suppositions may be made, of which our formulæ will give us the consequences.

16. We may suppose that no land is thrown out of cultivation in consequence of the tax: that the demand increases, and with it the price, so as to keep a_n in cultivation. That this may be so, we must have

$$pr_n - c_n q \text{ not less than } 0 \text{ by the second Axiom,}$$

$$\text{and } p'r_n - t_n - c_n q \text{ not less than } 0, \text{ for the same reason.}$$

Suppose that we take the soil a_n to be exactly of the *limiting quality*, so that by the 5th and 6th Axioms,

$$pr_n - c_n q = 0, \quad p'r_n - t_n - c_n q = 0, \quad \text{and let } t_n = k_n p' r_n,$$

$$\therefore \; p'r_n(1 - k_n) = c_n q, \quad pr_n = c_n q, \quad \therefore \; p'(1 - k_n) = p,$$

Here also $u = 0$, and $v = 0$, because no land is thrown out, therefore putting for p' its value, we have by the last Article

$$\text{Increase of price} \ldots\ldots = \frac{arp}{1-k_n} - arp = \frac{k_n arp}{1-k_n}.$$

$$\text{Diminution of rent} \ldots = arp - ar\frac{1-k}{1-k_n}p = arp \cdot \frac{k-k_n}{1-k_n}.$$

$$\text{Diminution of profits} = 0.$$

$$\text{Tax} \ldots\ldots\ldots\ldots\ldots = \frac{karp}{1-k_n}.$$

If the tax bear a given ratio to the produce for all soils, $k_n = k$, the diminution of rent is 0, the increase of price and the amount of the tax are equal, and the whole tax falls upon the consumers.

This is the case considered by the writers who follow Mr. Ricardo, and the conclusion depends entirely, as appears from the

investigation, on the supposition that the supply is perfectly unaffected by the tax.

It may be observed also, that it results from our formulæ, that if the poorer soils be taxed at a lower proportional rate than the average, the tax will so far fall on rent, and be taken from price. In this case, k_n is less than k. If the poorest soil pay no tax, the whole of the tax levied on other soils falls on rent, even on the supposition above-mentioned. In this case, $k_n = 0$.

17. Next let it be supposed, that there is no soil in cultivation of the limiting quality. Then the price will remain unaltered by the imposition of the tax, for the demand and the supply each remains unaltered, and therefore the price. The cost of production does not affect prices in this case, because there is nothing produced which barely pays the cost of production with profits. Therefore $p' = p$, $u = 0$, $v = 0$.

Diminution of rent $= at =$ whole tax.

In this case, the tax falls wholly on rent.

This is the case in which the produce of every acre, being more than sufficient to pay expenses and profits, and thus leaving a rent, there is yet no poorer land which can be taken into cultivation, and no method known of applying additional capital to old land, with diminished, but sufficient returns. And in this case, it appears that all taxes which are taken from the produce of the land, are ultimately paid entirely by the landlord.

18. Let us now return to the case in which there is a certain quality of land which only just pays expenses and profits, and which is therefore liable to be thrown out of cultivation by a tax, or if kept in cultivation, affects prices. Let a_n be the quantity of this land, and r_n its rate of produce, supposing it, as a sufficiently close approximation at present, to be of uniform

quality. Also let r_{n-1} be the rate of produce of the next quality, and a_{n-1} its quantity. When a_n is thrown out of cultivation, the produce diminishes from ar to $ar - a_n r_n$ or $ar(1-u)$: hence the price increases from p to p'. Suppose $p' = (1+w)p$, then since the increase of price depends on the diminution of supply, w is a *function* of u as has already been explained.

In the case in which limiting soils determine the price, since p is the limiting price when the lowest rate of produce is r_n, and p' when it is r_{n-1}, with a tax t_{n-1}: we shall have by Axioms 5 and 6, putting $t_{n-1} = k_{n-1}\, p' r_{n-1}$,

$$p' r_{n-1}(1 - k_{n-1}) = q c_{n-1},$$
$$p r_n = q c_n,$$

and dividing, observing that $\dfrac{p'}{p} = 1 + w$,

$$\therefore\ (1+w)(1-k_{n-1})\frac{r_{n-1}}{r_n} = \frac{c_{n-1}}{c_n} \ \ldots\ldots\ldots (a),$$

the quantities $\frac{r_{n-1}}{r_n}$ and $\frac{c_{n-1}}{c_n}$ depend upon u, the diminution of production, if we suppose the scale of soils, that is, the fertility and quantity of each quality, and the capital requisite for its cultivation, to be known. On this supposition, all the quantities in the preceding equation (a) would be functions of u, and hence the equation would serve to determine u, and we should find the quantity by which the produce was diminished.

But without attempting this exact solution of the problem, we shall be able to obtain approximations sufficiently general and accurate.

If we suppose the capital employed on an acre each of the last two qualities to be the same, and the produce only to be different in the ratio $1 + \rho : 1$, we have

$$(1+w)(1-k_{n-1})(1+\rho) = 1.$$

And without this last supposition, if $1+\rho = \frac{r_{n-1}}{c_{n-1}} \frac{c_n}{r_n}$ = the ratio of the rate of the produce to the capital on the two last soils, we shall still have

$$(1+w)(1-k_{n-1})(1+\rho) = 1,$$

$$\therefore\ p' = p(1+w) = \frac{p}{(1-k_{n-1})(1+\rho)},$$

$$\text{and if } k_{n-1} = k,\ (1+w)(1-k)(1+\rho) = 1 \ldots\ldots\ldots\ldots (b).$$

By Axiom 4, we shall in general assume w to be a multiple of u, so that $w = eu$; and $p' = (1+eu)\,p$. The reasons have been given for supposing e to be about 3.

If we suppose the price to be inversely as the supply, according to the hypothesis mentioned in Art. 9, we have

$$\frac{p'}{p} = \frac{1}{1-u},\quad \therefore\ 1+w = \frac{1}{1-u} \ldots\ldots\ldots\ldots\ldots\ldots (c).$$

19. For the sake of abbreviation let R represent the whole rent, Q the whole of the return to capital with profit, P the whole price, T the whole tax: and let τR, τQ, τP represent the quantities by which, in consequence of the tax, R, Q, P are increased; which quantities therefore will be negative if those totals are diminished, then, as in Art. 15,

$$\left.\begin{aligned} -\tau R &= arp - ar(1-u)(1-k)p' - acqv \\ -\tau Q &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots acqv \\ \tau P &= \ldots\ldots\ ar(1-u)p' - arp \\ T &= \ldots\ldots\ kar(1-u)p' \end{aligned}\right\} \ldots\ldots (A).$$

Putting here for p' its value $(1+w)\,p$ we have

$$\left.\begin{aligned} -\tau R &= arp\,\{1-(1-u)(1+w)(1-k)\} - acqv \\ -\tau Q &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots acqv \\ \tau P &= arp\,\{(1-u)(1+w)-1\} \\ T &= arpk\,(1-u)(1+w) \end{aligned}\right\} \ldots\ldots (B).$$

Also $-{}_{T}R = arpk\,(1-u)\,(1+w) - arp\,(w-u-uw) - acqv$.

Hence the tax falls on rent, except the portion $arp\,(w-u-uw)$ which falls in price, and $acqv$ which falls on profit.

20. Let price vary inversely as supply, and we have as before by (c), Art. 18,

$\frac{p'}{p} = \frac{1}{1-u}$, or $(1+w)\,(1-u) = 1$. Hence equations (*B*) give us

$$\begin{aligned} -{}_{T}R &= arpk - acqv \\ -{}_{T}Q &= acqv \\ {}_{T}P &= 0 \\ T &= arpk. \end{aligned}$$

Therefore in this case the consumer pays no more than he did before the tax. The tax is taken entirely from rent and profits. The consumer however receives less for his money.

21. Let the price be determined as in Axiom 5, by the expense of production on the limiting soil, both before and after the tax is imposed: a_n, and a_{n-1} being the limiting soils in the two cases. Therefore by equation (*b*), Art. 18,

$$(1+w)\,(1+\rho)\,(1-k) = 1,\quad 1+w = \frac{1}{(1+\rho)\,(1-k)},$$

and by the substitution of this value equation (*B*) becomes

$$\left.\begin{aligned} -{}_{T}R &= arp\left\{1 - \frac{1-u}{1+\rho}\right\} - acqv \\ -{}_{T}Q &= acqv \\ {}_{T}P &= arp\left\{\frac{1-u}{(1-k)\,(1+\rho)} - 1\right\} \\ T &= arpk\,\frac{(1-u)}{(1-k)\,(1+\rho)} \end{aligned}\right\} \ldots\ldots\ldots (C).$$

$$\text{Hence,}\quad -{}_{T}R = arp\,\frac{\rho+u}{1+\rho} - acqv,$$

$$ {}_{T}P = arp\,\frac{k-\rho-u+k\rho}{(1-k)(1+\rho)}.$$

22. We shall now suppose the average rate of produce r to be a known multiple of the lowest rate of produce r_n; (it may perhaps in this country be two or three times the latter.) Generally let $r=mr_n$, and since $uar=a_n r_n$, we have $mua=a_n$, or $mu=\frac{a_n}{a}$.

Also the quantity of the last soil a_n will depend upon the interval which it is found necessary to take between its fertility and that of the next quality a_{n-1}. If, in order to reduce the problem to calculation, we suppose the worst soil (including all from rate r_n, to rate r_{n-1}) to consist of qualities improving by insensible shades, and of each of which there is an equal quantity, we can easily express this dependence: and this supposition must bring us very near the truth. Let the rate at which these worst qualities of soil improve be such, that if the same rate of improvement continued for the whole quantity of land in the country (which is a), the best soil would yield a produce r'. Let this be called the *hypothetical greatest rate of produce.* It may probably not differ much from r, the actual greatest rate of produce. But let generally $r'=\mu r_n$, or let the hypothetical greatest rate be μ times the least. Then we shall have, since the rate of improvement is uniform,

$$\frac{r_{n-1}-r_n}{r'-r_n}=\frac{a_n}{a}, \text{ or } \frac{r_{n-1}-r_n}{(\mu-1)\,r_n}=\frac{a_n}{a},$$

$$\therefore \frac{r_{n-1}}{r_n}=1+(\mu-1)\frac{a_n}{a}. \quad \text{And as before } \frac{a_n}{a}=mu,$$

$$\therefore \frac{r_{n-1}}{r_n}=1+m\,(\mu-1)\,u. \quad \text{But } \frac{r_{n-1}}{r_n}=1+\rho, \text{ (if } c_{n-1}=c_n);$$

$$\therefore \rho=m\,(\mu-1)\,u.$$

Also if in equation (b) we put for w its value eu, we have an equation which expresses that the price arising from dimi-

nished supply is the same as the price of production on the new limiting soil. We obtain thus

$$\frac{1}{(1+\rho)(1-k)} = 1 + eu.$$

23. Substituting in equations (*C*), the values

$$\frac{1}{(1+\rho)(1-k)} = 1 + eu, \quad \rho = m(\mu - 1)\,u, \text{ we have}$$

$$\left.\begin{aligned} -\tau R &= arp\,\frac{\{1+m(\mu-1)\}\,u}{1+m(\mu-1)\,u} - acqv \\ -\tau Q &= acqv \\ \tau P &= arp\,\{(e-1)\,u - eu^2\} \\ T &= arpk\,(1+eu)(1-u) \end{aligned}\right\} \ldots\ldots\ldots (D).$$

Also by the equation

$$\frac{1}{1-k} = (1+eu)\,\{1+m(\mu-1)\,u\} \ldots\ldots (d),$$

u is known when k is given, and *vice versa*.

24. We may obtain also the relation between u and v. For we have

$$pr_n = qc_n, \quad pa_n r_n = qa_n c_n, \quad puar = qvac.$$

$$\text{Hence,} \quad aqcv = apru.$$

Substituting this in equations (*D*), we have

$$-\tau R = arpu\,m(\mu-1)\,\frac{1-u}{1+m(\mu-1)\,u}$$

$$-\tau Q = arpu.$$

25. When u is small we may simplify the expressions.

Since a_n was thrown out of cultivation by the imposing of the tax, it is manifest by Axiom 2, that we have

$$p'(1-k)\,r_n < c_n q; \quad \therefore\ r_n < \frac{c_n q}{p'(1-k)}.$$

$$\text{But } pr_n = c_n q; \;\therefore r_n = \frac{c_n q}{p};$$

$$\therefore p > p'(1-k), \text{ or } > p(1+w)(1-k),$$

$$\text{and } (1+w)(1-k) < 1.$$

If the tax be not a very large fraction, for instance, if k be $\frac{1}{10}$, or less, we may neglect its powers;

$$\therefore 1+w < \frac{1}{1-k}, \text{ or } < 1+k; \;\therefore w < k,$$

hence also we may neglect the powers of w, and the products wk, &c. Also by equation (b), we have generally

$$(1+w)(1-k) = \frac{1}{1+\rho}$$

$$1+\rho = \frac{1}{(1+w)(1-k)} = 1+k-w, \text{ nearly};$$

$\therefore \rho = k - w$, neglecting powers of k, w; and therefore ρ is also small.

Also we have supposed by Axiom 4, w to be $= eu$, and hence u also is small.

26. Expanding equation (d), Art. 23, we have

$$1 + \{e+m(\mu-1)u\} + em(\mu-1)u^2 = \frac{1}{1-k}; \text{ or, putting } \frac{k}{1-k} = f,$$

$$u^2 + \frac{e+m(\mu-1)}{em(\mu-1)}u = \frac{f}{em(\mu-1)}.$$

And hence

$$u + \frac{e+m(\mu-1)}{2em(\mu-1)} = \pm\frac{e+m(\mu-1)}{2em(\mu-1)}\left\{1+\frac{4fem(\mu-1)}{\{e+m(\mu-1)\}^2}\right\}^{\frac{1}{2}},$$

expanding to three terms of the binomial

$$= \pm\frac{e+m(\mu-1)}{2em(\mu-1)} \pm \frac{f}{e+m(\mu-1)} \mp \frac{f^2 em(\mu-1)}{\{e+m(\mu-1)\}^3},$$

and taking the upper sign

$$u = \frac{f}{e+m(\mu-1)} - \frac{f^2 em(\mu-1)}{\{e+m(\mu-1)\}^3}.$$

If $k = \frac{1}{10}$, $e=3$, $m(\mu-1) = 12$; we shall have $f = \frac{1}{9}$, and the last term in the above expression is $-\frac{4}{30375}$ which may be neglected without error.

$$\text{Hence,} \quad u = \frac{f}{e+m(\mu-1)},$$

which is the result that we should have obtained from equation (d) by omitting u^2.

27. Substituting this value, and omitting u^2, &c. equations (D) become

$$-\tau R = arpf\frac{1+m(\mu-1)}{e+m(\mu-1)} - acqv$$
$$-\tau Q = acqv, \text{ \&c.}$$

or putting $apru$ for $aqcv$

$$\left.\begin{aligned} -\tau R &= arpf.\frac{m(\mu-1)}{1+m(\mu-1)} \\ -\tau Q &= arpf.\frac{1}{1+m(\mu-1)} \\ \tau P &= arpf.\frac{e-1}{1-m(\mu-1)} \\ T &= arpf \end{aligned}\right\} \ldots\ldots\ldots\ldots (E).$$

Hence, $\tau P - \tau Q - \tau R = T$, as it should be.

28. Hence the portions of the tax which fall on rent, profits and prices, are as $m(\mu-1)$, 1, $e-1$, respectively.

Thus if the average produce be four times, and the greatest produce seven times, that which pays mere profits (e being 3), we have $m=4$, $\mu=7$; and the portions which fall on rent, profits, and prices will be respectively as 24, 1, 2: on these suppositions less than $\frac{1}{13}$ the tax falls on prices.

If in this case the tax be $\frac{1}{10}$ the produce,

$$k = \frac{1}{10},\ f = \frac{1}{9},\ u = \frac{1}{9 \times 27} = \frac{1}{243},\ w = \frac{1}{81},$$

prices are raised by $\frac{1}{81}$ of their amount, and the supply is diminished by $\frac{1}{243}$.

If the average produce be 3 times, the greatest 5 times the rent, supply is diminished by $(u =) \frac{1}{135}$, prices rise $(w =) \frac{1}{45}$, the consumer pays $\frac{2}{15}$, the landlord $\frac{12}{15} = \frac{4}{5}$ of the tax.

If the average produce be only $1\frac{1}{2}$ times the least, and the greatest produce twice the least; $u = \frac{2}{81}$, $w = \frac{2}{27}$, price pays $\frac{4}{9}$, rent $\frac{4}{9}$.

This is a case where the different parts of the land, and the capital employed on it, are nearly of uniform value; and where nevertheless the cultivation is bounded by the smallness of the demand; a state of things which would probably not be permanent.

29. If the average rent of the country be a given portion of the produce, we can determine m. Let the average capital on an acre be l times the capital on an acre of the lowest soil. Hence $c = lc_n$, and $r = mr_n$. Now $pr_n = qc_n$, and $plmr_n = qmlc_n$, whence $plr = qmc$. And the whole rent $= arp - acq = arp\left(1 - \frac{l}{m}\right)$, the produce being arp. Hence the produce is of the rent a multiple $\frac{m}{m-l}$. Thus if the average rent be $\frac{1}{2}$ the produce, $\frac{m}{m-l} = 2$; $m = 2l$. If the average rent be $\frac{1}{3}$ the produce, $m = \frac{3l}{2}$; if rent $= \frac{1}{4}$ produce, $m = \frac{4l}{3}$. And if we suppose the capital employed on good land to be 2, 3, 4, &c. times that which is employed on the same quantity of very poor land, we shall have the corresponding

values of m. Thus if the average capital on the soil be three times that which is employed on the worst land, $l=3$; and if the average rent be $\frac{1}{4}$ of the produce, this gives $m = \frac{4f}{3} = 4$. It is to be observed that the capital here supposed to be employed on good land, includes all the successive doses up to the last, which are introduced in Mr. Mill's mode of viewing the subject; and therefore in a highly cultivated country l would perhaps not be less than 3 or 4, and might probably be much more. And it may be noticed also that in the case in which the charges now under consideration operate by withdrawing the last dose of capital from cultivated land, l is the multiple which the whole capital employed on average land is of this last dose.

It appears then that if capital be accumulated on the land, l will be greater than 1; perhaps as great as 2, 3 or 4. That in this case m will always be greater than l, and μ greater than m; and that then the fraction of a tax on produce which falls on price will be very small, the main portion falling on rent.

It is to be observed, that the values of m are quite independent of the actual quantity of the produce. The produce may be greatest when m is least, because it may be that a large quantity of capital is employed on *every* soil, and the produce, though very great, may be principally employed in paying the profits of capital.

30. Such are the results of the principles which were originally assumed, when traced to their consequences in the final distribution of a tax, on the suppositions made in the course of the investigation. Some of these suppositions have been arbitrary, but the nature of the results would not be affected, and their quantity but slightly modified, by reasoning on any other admissible hypothesis. Thus it has been supposed, in Art. 22, that $(c_{n-1} = c_n)$ the ratio of the productive powers of the two

lowest soils is properly estimated by assuming the same capital to be applied to an acre of each, and comparing the results, whereas it may be that from the nature of the agriculture of these soils the same capital is not applied to each. Also it is assumed in Art. 22, that if we have among the lowest cultivated soils, 10000 acres which gives an excess of not more than 6 per acre above the least produce, we shall have 20000 acres which gives an excess of not more than 12, and so on proportionally. These are assumptions made to reduce the problem to mathematical definiteness and regularity. But their inaccuracy will affect the result only by quantities much less than those which we investigate. The errors will be changes of the second order, or often of lower orders still. In the same manner the assumption is introduced that there is a limiting soil which pays exactly profits and nothing more. But if we suppose that though there may not be exactly any such soil, the soils above this limit will be cultivated, and those below it will not, we come to nearly the same conclusions.

31. Let us now consider the effect of a *land-tax:* and let the tax be a given sum (t) per acre. Resuming the equations of Art. 15, we have (a_n being thrown out of cultivation by the tax),

$$-{}_{\tau}R = arp - ar(1-u)p' + (a-a_n)t - acqv$$

$$-{}_{\tau}Q = acqv,$$

$$_{\tau}P = ar(1-u)p' - arp,$$

$$T = (a-a_n)t,$$

as before, let $p' = (1+w)p$, and we have

$$-{}_{\tau}R = arp\{1-(1+w)(1-u)\} + (a-a_n)t - acqv.$$

Also for the limiting soils with and without taxes respectively, we have

$$p'r_{n-1} - t_{n-1} = c_{n-1}q,$$

$$pr_n = c_n q; \text{ and eliminating } q.$$

$$\frac{p' r_{n-1}}{c_{n-1}} - \frac{t}{c_{n-1}} = \frac{p r_n}{c_n}, \text{ whence } \frac{p'}{p} \frac{r_{n-1} c_n}{r_n c_{n-1}} = 1 + \frac{c_n t}{c_{n-1} p r_n}.$$

Let as before

$$\frac{r_{n-1} c_n}{r_n c_{n-1}} = 1 + \rho, \quad \frac{c_n}{c_{n-1}} = 1, \quad r = m r_n; \text{ and we have}$$

$$(1 + w)(1 + \rho) = 1 + \frac{mt}{pr}.$$

On the same suppositions as those already made in Art. 22, we shall have, as there,

$$\rho = m(\mu - 1) u, \quad w = eu; \text{ whence}$$

$$(1 + eu)\{1 + m(\mu - 1) u\} = 1 + \frac{mt}{pr};$$

and hence, as before, omitting u^2, &c.

$$u = \frac{mt}{pr\{e + m(\mu - 1)\}}, \text{ or if } \frac{t}{pr} = k,$$

$$u = \frac{mk}{e + m(\mu - 1)},$$

k is the fraction when the tax t is of the *average* produce.

Also as above, $cqv = rpu$, $a_n = mua$.

Substituting these values in the expression above given, and also w for eu, and kpr for t, we have

$$\begin{aligned} -\tau R &= -arp\{(e - 1) u - eu^2\} + arpk(1 - mu) - arpu \\ &= arpk(1 - mu) - arpeu(1 - u) \\ &= arp\{k - eu - (mk - eu) u\}. \end{aligned}$$

And taking the value of u above found, this gives

$$-\tau R = arpk\left\{\frac{m(\mu - 1) - e(m - 1)}{e + m(\mu - 1)} - \frac{m^2(\mu - 1)k}{\{e + m(\mu - 1)\}^2}\right\}.$$

If, as before, we suppose $e = 3$, $m = 4$, $\mu = 7$, the two terms within the brackets become $\frac{5}{9}$ and $\frac{384k}{729}$. If k be small, as $\frac{1}{10}$, the latter term is small in comparison of the first.

$$\text{Also } -\tau Q = acqv = arpu = arpk\frac{m}{e + m(\mu - 1)},$$

$$\tau P = arp\{(1 + w)(1 + u) - 1\}$$
$$= arp\{(e-1)u - eu^2\}$$
$$= arpk.\left\{\frac{(e-1)m}{e+m(\mu-1)} - \frac{em^2k}{\{e+m(\mu-1)\}^2}\right\}.$$

Hence the parts of the tax, which affect rent, profits and price, are nearly as $m(\mu-1) - e(m-1)$, m, and $(e-1)m$ respectively.

In the case where m, μ, e have the values above-mentioned, these parts are as 15, 4, 8.

In general, the part which falls on rent will be the greatest.

32. Let the tax be a certain portion of the rent, viz. a fraction s of the rent which arose when there was no tax;

$$\therefore\ t_n = s(pr_n - qc_n)$$

$$\therefore \text{ the rent under the tax} = pr_n - s(pr_n - qc_n) - qc_n \text{ by Ax. 6.}$$
$$= (1-s)(pr_n - qc_n).$$

In this case the tax falls wholly upon rent.

33. Let the tax be a *tax on profits*, viz. a fraction s of the previous profit. Therefore

$$t_m = s(q-1)c_m, \text{ and } a_m t_m = s(q-1)a_m c_m.$$

And if t be the average tax, a_n the land thrown out of cultivation by it,

$$\text{the whole tax} = (a - a_n)t = s(q-1)(ac - a_n c_n),$$

and hence, by Article 15,

$$-\tau R = arp - ar(1-u)p' + s(q-1)(ac - a_n c_n) - acqv.$$

$$\text{As before, } p' = (1+w)p,\ a_n c_n = vac;$$

$$\text{and hence, since } acqv = arpu,$$

$$-\tau R = arp\{1 - (1-u)(1+w)\} + asc(q-1)(1-v) - acqv$$

$$-\tau R = ac(q-1)s - acv(q-1)s - arpw(1-u)$$

$$= ac(q-1)s - arpw - arp\left\{\frac{(q-1)su}{q} - wu\right\}.$$

The two first terms are the most considerable.

Using the equations of price, according to Axioms 5 and 6, both with and without the tax, we have

$$p' r_{n-1} = c_{n-1} q + s(q-1) c_{n-1},$$

$$p r_n = c_n q.$$

Whence $\dfrac{p' r_{n-1} c_n}{p r_n c_{n-1}} = 1 + \dfrac{(q-1)s}{q}$; or using w and ρ as before,

$$(1+w)(1+\rho) = 1 + \frac{(q-1)s}{q}.$$

Also as before, $w = eu$, $\rho = m(\mu-1)u$, and, therefore, omitting u^2, &c.

$$eu + m(\mu-1)u = \frac{(q-1)s}{q},$$

$$u = \frac{(q-1)s}{q\{e+m(\mu-1)\}}, \quad w = \frac{(q-1)es}{q\{e+m(\mu-1)\}}.$$

Putting for w its value in $-\tau R$, and omitting the latter term which involves s^2

$$-\tau R = ac(q-1)s - arp\frac{(q-1)s}{q}\,\frac{e}{e+m(\mu-1)}.$$

or since $ac = \dfrac{arpu}{qv}$

$$-\tau R = arp\frac{(q-1)s}{q}\left\{\frac{u}{v} - \frac{e}{e+m(\mu-1)}\right\}.$$

Also, since the effect of the tax cannot diminish the rate of profits, by Axiom 6,

$$-\tau Q = acqv = arpu = arp\frac{(q-1)s}{q}\,\frac{1}{e+m(\mu-1)},$$

$$\tau P = arp\{(1+w)(1-u)-1\} = arp\frac{(q-1)s}{q}\,\frac{e-1}{e+m(\mu-1)} - arpeu^2.$$

The whole tax $T = ac(q-1)s = arp\frac{(q-1)s}{q}\cdot\frac{u}{v}$;

and thus the portions of it which fall on rent, capital with profits and price, have been obtained.

The fraction $\frac{u}{v}$ is known if the diminution of the capital employed, and of the produce obtained, be known with reference to the whole.

Or thus. Let the *average* capital employed on an acre, be l times the capital employed on the worst soil.

$$\text{Hence, } c = lc_n. \text{ Also } mua = a_n,$$

$$\therefore muac = la_nc_n = lvac, \text{ whence } \frac{u}{v} = \frac{l}{m}.$$

Let $e = 3$, $m = 4$, $\mu = 7$ as before, and let $l = 2$. Then the portions of the tax which fall on rent, capital and price, will be respectively as 21, 2, 4.

In this case also much the largest portion falls on rent.

If profits be 10 per cent., $q = \frac{11}{10}$, $\frac{q-1}{q} = \frac{1}{11}$. And if the tax be 20 per cent. of the profits, $s = \frac{1}{5}$. Therefore we shall have

$$u = \frac{\frac{1}{11}\cdot\frac{1}{5}}{27} = \frac{1}{1485}, \quad w = 3u = \frac{1}{495}.$$

Mr. Thompson in his Theory of Rent, p. 33, supposes a case of such profits and tax, with a diminution of produce of $\frac{1}{165}$ nearly, and of capital employed of $\frac{1}{110}$ nearly. Hence, taking these numbers

$$u = \frac{1}{165} = \frac{1}{55}\,\frac{1}{e+m(\mu-1)}; \;\therefore e+m(\mu-1)=3; \text{ and } \frac{u}{v} = \frac{110}{165} = \frac{2}{3}.$$

Also the price is supposed to increase by

$$\frac{1}{110} = w; \therefore e = \frac{w}{u} = \frac{165}{110} = \frac{3}{2}.$$

Hence $m(\mu - 1) = \frac{3}{2}$, which is satisfied by assuming $\mu = 2$, $m = \frac{3}{2}$.

In this case the portions of the tax which fall on rent, capital and price, are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. The whole tax is of the original produce a fraction $\frac{(q-1)s}{q} \cdot \frac{u}{v} = \frac{1}{55} \cdot \frac{2}{3} = \frac{2}{165}$: and the part which falls on rent is $\frac{1}{330}$ of the produce.

Also the whole return to capital $acq = \frac{2}{3}$ the whole produce, and the rent $= \frac{1}{3}$ the produce at first.

If $e = 2$, $m = 2$, $\mu = 3$, $\frac{u}{v} = \frac{1}{2}$, the portions of the tax which fall on rent, capital and price are equal.

35. Similar reasoning may be applied to determine the result of any other changes, for instance, of improvements in agriculture, such as those which Mr. Thompson considers, p. 12, &c. The conditions of the change being known, it may be ascertained how far its final consequence will be to increase rent, and how far to produce other effects. The method will appear in the following cases.

Let the improvement be "one which makes a *saving* in bringing some part of the produce to market, as, for instance, a threshing machine."

In this case we have to suppose a diminution of c, or at least of c_n, r_n and r remaining. Let c_n become c'_n; and suppose that $pr_n < c_n q$, so that without the improvement a_n could not be cultivated, and that $p'r_n =$ or $> c'_n q$, so that the improvement

brings this quality of land into cultivation. The rent will be changed from

$$apr - acq$$

$$\text{to } ap'r + p'a_n r_n - ac'q - qa_n c'_n.$$

(*N.B.* In this case a, ar, and ac do not include a_n.)

$$\text{Let } c' = c(1-y), \quad c'_n = c_n(1-y_n).$$

Also let $a_n r_n = uar$, $a_n c_n = vac$, $p' = (1-w)p$, as before.

Hence the rent increases from $apr - acq$, to

$$apr(1-w) + apru(1-w) - acq(1-y) - acqv(1-y_n);$$

the increase is

$$apr(u - w - uw) + acq\{y - v(1-y_n)\}.$$

The rent is increased by the increased produce, and diminished by the fall of price.

The conditions of price before and after the improvement, give us

$$pr_{n-1} = c_{n-1}q, \quad p'r_n = c'_n q,$$

$$\therefore \frac{p'}{p} = \frac{c'_n r_{n-1}}{c_{n-1} r_n} = \frac{c'_n}{c_n} \cdot \frac{c_n r_{n-1}}{c_{n-1} r_n}; \text{ or if } \frac{c_n r_{n-1}}{c_{n-1} r_n} = 1 + \rho,$$

$$1 - w = (1 - y_n)(1 + \rho).$$

Let the same suppositions be made as before in Art. 22, respecting the relation between the produce and the quantity of the lowest soil. Then we shall have

$$\rho = m(\mu - 1)u. \quad \text{Also let } w = eu,$$

$$\therefore \frac{1 - eu}{1 + m(\mu-1)u} = 1 - y_n, \text{ and omitting } u^2, \&\text{c., } u = \frac{y_n}{e + m(\mu - 1)}.$$

Also as in the last Article,

$$acq = apr\frac{u}{v}; \text{ whence the increase of rent becomes}$$

$$apr\left\{\frac{uy}{v} - u(1-y_n) - w + u - uw\right\}$$

$$= arp\left\{\frac{uy}{v} - w - (w - y_n)\,u\right\},$$

$$= apr\left\{\frac{uy}{v} - \frac{ey_n}{e + m(\mu - 1)} - m(\mu - 1)u^2\right\};$$

the last term is of the order y^2.

If we suppose the diminution of the capital requisite, to be the same on the lowest soil as on the average, $y_n = y$, and, neglecting y^2,

$$\text{increase of rent} = apry\left\{\frac{u}{v} - \frac{e}{e + m(\mu - 1)}\right\}.$$

For $\frac{u}{v}$, we may put $\frac{l}{m}$ as in the last article.

We have similarly,

$$\text{increase of profits} = acqv = apru = apry\,\frac{1}{e + m(\mu - 1)};$$

$$\text{diminution of price} = apr - apr\,(1 + u)\,(1 - w)$$

$$= apry\,\frac{e - 1}{e + m(\mu - 1)},\ \text{omitting } apr\,.\,eu^2.$$

The cases in which the improvement would go principally to increase rents, will be the same as those in which a tax on profits goes principally to diminish rents, as in the last Article.

36. Let the improvement be "one which requires an *increased outlay* upon the land, but returns an increased quantity of produce, as, for instance, drill husbandry."

Let c, c_n, r, r_n become c', c'_n, r', r'_n: and let a_n come into cultivation under the improved system. Then, as before, by the conditions of price

$$pr_{n-1} = qc_{n-1},\quad p'r'_n = qc'_n,$$

$$\frac{p'}{p} = \frac{c'_n r_{n-1}}{c_{n-1} r'_n}.$$

And if for the sake of simplification, we suppose, as in the last Article,

$$\frac{p'}{p} = 1 - w,\ \frac{c_n r_{n-1}}{c_{n-1} r_n} = 1 + \rho,\ \text{and make}$$

$\frac{c'_n}{c_n} = 1 + y_n$, $\frac{r'_n}{r_n} = 1 + z_n$, $\frac{c'}{c} = 1 + y$, $\frac{r'}{r} = 1 + z$, we have

$$1 - w = \frac{(1+\rho)(1+y_n)}{1+z_n}.$$

The increase of quantity in this case is

$$ar' + a_n r'_n - ar = ar\{z + u(1+z_n)\}.$$

And hence by Axiom 4, we shall have

$$w = e(z + u + uz_n);$$

$$\therefore \{1 - e(z+u+uz_n)\}(1+z_n) = (1+\rho)(1+y_n).$$

Also the same suppositions being made as before, in Art. 22, with respect to the limiting soil, $\rho = m(\mu - 1)u$. Hence, omitting quantities of the order y^2, we have

$$u = \frac{z_n - y_n - ez}{e + m(\mu - 1)}.$$

If the increase of capital and produce be the same on the limiting soils as on the average,

$$z_n = z,\ \ y_n = y,\ \ u = -\frac{(e-1)z + y}{e + m(\mu - 1)}.$$

This is negative. In general, therefore, with such improvements, bad land would be thrown out of cultivation, in consequence of the increased produce from good land. But in the course of time, the demand would probably receive a permanent augmentation, which would counteract this effect.

Here $w = e(z + u)$ nearly, $= e\,\frac{m(\mu - 1)z + z - y}{e + m(\mu - 1)}$.

Now the rent increases from $apr - acq$ to

$$ap'r' + p'a_n r'_n - aqc' - qa_n c'_n$$

$$= apr(1-w)(1+z) + apru(1-w)(1+z_n) - aqc(1+y) - aqcv(1+y_n).$$

Hence the increase of rent, omitting terms involving

$$wu, \ z_n u, \ y_n v, \text{ \&c. is}$$

$$= apr(z - y - w + u) - acqv,$$

$$(\text{since } acqv = apru,) = apr(z - y - w).$$

Putting for w its value, this is

$$= apr \cdot \frac{m(\mu-1)(z-y-ez)}{e+m(\mu-1)} = - apr \frac{m(\mu-1)\{(e-1)z-y\}}{e+m(\mu-1)},$$

in this case we have a diminution also of the rent, till the increased demand comes into action.

The increase of return and profit is

$$qac' + qa_n c_n' - qac = qacy + acqv(1+y_n)$$

$$= aqcy + apru, \text{ omitting } vy_n.$$

If we suppose the demand to have increased, so that with the new system of production prices are the same as before, we shall have the rent increased from $arp - acq$ to

$$ar'p + pa_n r'_n - ac'q - qa_n c_n'.$$

Therefore the increase is

$$arpz - acqy + apru(1 + z_n) - aqcv(1 + y_n),$$

$$\text{also } apru = aqcv,$$

and omitting the smaller terms, the increase of rent is

$$arpz - acqy,$$

which is the whole increase of produce minus the return and profit on the increased outlay.

W. WHEWELL.

TRINITY COLLEGE,
March 30, 1829.

Note on Art. 2.

Let a tax be imposed on wages which is a fraction k of the whole. Let the labourer's consumption of manufactured goods be a fraction m of his whole consumption; and let the portion of the value of goods which depends on wages be a fraction n of the whole. Let x be the fraction by which wages rise in consequence of the tax. Now goods before the tax consist in value of m wages, and $1-m$ raw produce. When wages are increased in the ratio $1+x:1$, the value of goods will therefore be increased in the ratio $(1+x)m+1-m:1$, or $1+mx:1$. Also before the tax the labourer's consumption is n in goods and $1-n$ in produce. And if it continue the same in each article, (which is supposed) its parts will be $n(1+mx)$ and $1-n$: and these, together with the tax, which is $k(1+x)$ make up the whole new wages. Therefore

$$n(1+mx)+1-n+k(1+x)=1+x;$$

$$\therefore\ k=x-kx-mnx,\quad x=\frac{k}{1-k-mn},$$

and the effect on the price of goods $= mx = \dfrac{mk}{1-k-mn}$.

$$\text{If } k=\frac{1}{10},\quad m=\frac{1}{2},\quad n=\frac{1}{2},\quad x=\frac{4}{26}=\frac{2}{13},\quad mx=\frac{1}{13}:$$

by a tax on wages of $\frac{1}{10}$, wages rise $\frac{2}{13}$, and prices $\frac{1}{13}$.

The result given in Art. 2, is inaccurate on the supposition of a tax of one tenth on wages; for the *first* step of the effect of such a tax would be to increase wages by one *ninth,* in order that when one tenth of the increased wages was deducted, the part remaining to the labourer, might be the same as before.

It is to be noticed, that the reasoning of the preceding pages differs from that of Mr. Thompson's Theory of Rent, only in the introduction of mathematical processes.

CORRECTIONS.

Page 192, line 5 *for* or *read* and.

.... 18 I have accidentally misstated Mr. Ricardo's object in the argument which I have quoted. The purpose which he has in view when he adduces it, is to shew that taxes on wages must fall on profits. This mistake however does not affect the use of the argument for illustrating the observation which I have founded upon it.

Page 192, line 23 *for* limits *read* limit.

Page 193, line 6 *for* informs *read* inform.

.... 8 *for* $\frac{4}{39}$ *read* $\frac{4}{30}$.

.... 9 *for* $\frac{2}{39}$ *read* $\frac{2}{30}$.

...... 195 ... 8 *for* And this *read* And it.

...... 196 ... 25 dele comma after *one*.

...... 198 ... 16 dele comma after *superimposed*.

...... 199, Axiom 3. The object of this Axiom is to assert that there will always be a limiting soil; and the clauses which precede this assertion are not necessarily and universally true. It might happen from some cause, for instance, from improvements in agriculture, that the produce of land should increase, and yet that the same soil as before should be the limiting soil; no new land being taken into cultivation. This correction of what is asserted, will not affect its application in any of the cases where it is introduced into the calculation; since the only use of the Axiom in the investigations is to furnish the equation which represents the condition of there being a limiting soil.

Page 201, Axiom 4. This Axiom comprehends only *one* of the elements of a change of price, the alteration of the supply. Any causes which affect demand directly, affect price through it; but such causes are not here considered.

Page 205, line 13 *for* profits *read* return to capital with profits.

In several places the word profits is used instead of the return to capital with profits. The reader who attends to the reasoning will easily make this correction.

Page 205, line 18 *for* Taxes *read* Partial Taxes.

Page 207, line 9, &c. Instead of $(a - a_n)t$, we ought to have in the formulæ $at - a_n t_n$.

This correction will not affect what follows.

Page 212, line 3, the part $acqv$ of the tax is said "to fall on profits." Agreeably to what has just been said, the expression should have been that it falls on the returns to capital. But even with this correction of the phrase, this portion of the tax cannot be considered as *lost* to the capitalist, because the diminution of the return here spoken of, arises from the capital being no longer employed in the same way. The term $acqv$ may be conceived to be compensated to the capitalist by some new employment of the displaced capital. But it was necessary to give some explanation of this term, and so far as agriculture is concerned, the description which I have given of it suggests the true nature of the alteration.

Page 213, line 2. The last soil is supposed to have less capital employed upon it than the richer ones; which will be true if the richer soils are capable of having dose after dose employed upon them, till we come to a dose which gives a return the same as the return of the poorer soils; and this is the theoretical supposition. If however the richer soils have less capital on them than the poorer ones, we shall have c less than c_n, and l in p. 217 will be less than 1. In this case the rent will be a considerable portion of the produce.

Page 213, line 17 *for* r *read* r_1.

..... 219, last line but one *for* t_{n-1} *read* t.

X. *On the Vowel Sounds, and on Reed Organ-Pipes.*

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[Read *Nov.* 24, 1828, and *March* 16, 1829.]

THE generality of writers who have treated on the vowel sounds appear never to have looked beyond the vocal organs for their origin. Apparently assuming the actual forms of these organs to be essential to their production, they have contented themselves with describing with minute precision the relative positions of the tongue, palate and teeth, peculiar to each vowel, or with giving accurate measurements of the corresponding separation of the lips, and of the tongue and uvula, considering vowels in fact more in the light of physiological functions of the human body than as a branch of acoustics.

Some attempts, it is true, have been made at various times to imitate by mechanical means the sounds of the human voice. Friar Bacon, Albertus Magnus, and others, are said to have constructed machines of this kind, but they were probably mere deceptions, like some contrivances which may be found in the works of Kircher and other writers of the same description*,

* Kircher. *Musurgia*, p. 303. Bp. Wilkins. *Dædalus*, p. 104. Schottus. *Mechanica. Hyd. Pneum.* p. 240, and *Magia Univ.* II. 155. B. Porta. *Magia Nat.* p. 287. The *Invisible Girl* was a contrivance of this kind. See *Nich. Journ.* 1802, p. 56, 1807, p. 69.

The abbè Mical (according to Rivarol)* made two colossal heads which were capable of pronouncing entire sentences, but the artist having destroyed them in a fit of disappointment at not receiving his expected reward from the government, and having left no trace of their construction, we are left completely in the dark, as to the means employed by him to produce the different sounds. He died about the year 1786. The only attempts which have a claim to a scientific character, are those of Kratzenstein and Kempelen; these gentlemen were both occupied about the year 1770, in the mechanical imitation of the voice, and have both in the most candid manner disclosed the means employed by them, and the results of their experiments, the first in a prize Essay presented to the Academy of Petersburgh in 1780†, the second in a separate treatise‡.

Kratzenstein's attempts were limited to the production of the vowels *a, e, o, u, i,* by means of a reed of a novel and ingenious construction attached to certain pipes, some of them of most grotesque and complicated figure, for which no reason is offered, save that experience had shewn these forms to be the best adapted to the production of the sounds in question.

Kempelen's treatise abounds with original and happy illustrations, and the author is no less remarkable for his ingenuity and success, than for the very lively and amusing way in which he has treated his subject. None of these writers, however, have succeeded in deducing any general principles.

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* Rivarol. *Discours sur l'universalitè de la langue françoise.* Borgnis. *Traitè des machines imitatives,* p. 160.

† The abstract of this Essay will be found in the *Act. Acad. Petrop.* for 1780, and the whole Essay in the *Journal de Physique,* Vol. XXI. See also Young's *Nat. Phil.* I. p. 783.

‡ *Le Mecanisme de la parole suivi de la description d'une Machine parlante, par M. de Kempelen. Vienne* 1791. Dr. Darwin must also be reckoned among the mechanical imitators of speech. See *Darwin's Temple of Nature* 1803, Note XI.

Kempelen's mistake, like that of every other writer on this subject, appears to lie in the tacit assumption, that every illustration is to be sought for in the form and action of the organs of speech themselves*, which, however paradoxical the assertion may appear, can never, I contend, lead to any accurate knowledge of the subject. It is admitted by these writers†, that the mouth and its apparatus, was constructed for other purposes besides the production of vowels, which appear to be merely an incidental use of it, every part of its structure being adapted to further the first great want of the creature, his nourishment. Besides, the vowels are mere affections of sound, which are not at all beyond the reach of human imitation in many ways, and not inseparably connected with the human organs, although they are most perfectly produced by them: just so, musical notes are formed in the larynx in the highest possible purity and perfection, and our best musical instruments offer mere humble imitations of them; but who ever dreamed of seeking from the larynx, an explanation of the laws by which musical notes are governed. These considerations soon induced me, upon entering on this investigation, to lay down a different plan of operations; namely, neglecting entirely the organs of speech, to determine, if possible, by experiments upon the usual acoustic instruments, what forms of cavities or other conditions, are essential to the production of these sounds, after which, by comparing these with the various positions of the human organs, it might be possible, not only to deduce the explanation and reason of their various positions, but to separate those parts and motions which are

* Kempelen's definition of a vowel, for instance, is deduced entirely from the organs of speech, "Une voyelle est donc un son de la voix qui est conduit par la langue aux lèvres, qui "le laissent sortir par leur ouverture. La différence d'une voyelle à l'autre n'est produite que "par le passage plus ou moins large que la langue ou les lèvres, ou bien ces deux parties "ensemble accordent à la voix." §. 106.

† Kempelen, §. 98.

destined for the performance of their other functions, from those which are immediately peculiar to speech (if such exist.)

In repeating experiments of this kind, it must always be kept in mind, that the difference between the vowels, depends entirely upon contrast*, and that they are therefore best distinguished by quick transitions from one to the other, and by not dwelling for any length of time upon any one of them. A simple trial will convince any person, that even in the human voice, if any given vowel be prolonged by singing, it soon becomes impossible to distinguish what vowel it is.

Vowels are quite a different affection of sound from both pitch and quality, and must be carefully distinguished from them. By quality, I mean that property of sound, by which we know the tone of a violin from that of a flute or of a trumpet. Thus we say, a man has a clear voice, a nasal voice, a thick voice, and yet his vowels are quite distinct from each other. Even a parrot, or Mr. Punch, in speaking, will produce A's, and O's, and E's, which are quite different in their *quality* from human vowels, but which are nevertheless distinctly A's, and O's, and E's. Again, as to pitch, all the vowels may be sung upon many notes of the scale, but of this more hereafter.

Euler † has discriminated these affections of sound, and dis-

* Kempelen has remarked this with his usual acuteness. Describing one of his early experiments, with a machine something like fig. 5, from which he obtained some of the vowels by covering its mouth with his left hand, he says, "J'obtins d'abord diverses voyelles, suivant "que j'ouvrois plus ou moins la main gauche. Mais cela n'arrivoit que lorsque je faisois "rapidement de suite divers mouvemens avec la main et les doigts. Lorsqu'au contraire "je conservois pendant quelque tems la même position quelconque de la main, il me paroissoit "que je n'entendois qu'un A. Je tirai bientôt de ceci la conséquence, que les sons de la "parole ne deviennent bien distincts que par la proportion qui existe entr'eux et qu'ils n'ob- "tiennent leur parfaite clarté que dans la liaison des mots entiers et des phrases." p. 400.

† *De motu aëris in tubis.* Schol. II. and III. Prop. 73. *Nov. Comm. Petrop.* XVI. and *Mem. Acad. Berlin* 1767, p. 354. See also Bacon *Hist. Nat.* §. 290.

tributed them among the different properties of the aerial pulsations as follows. The pitch depends on the number of vibrations in a given time. Loudness on the greater or less extent of the excursion of the particles. Quality and the vowel sounds, he thinks must depend on the form of the curve by which the law of density, and velocity in the pulse is defined, or upon the latitude of the pulse, but this he offers as a mere opinion, unsupported by experiment, save that* to account for the peculiar property of sound, by which we know a flute from a trumpet, &c. he remarks, that as the vibrations of each instrument are excited in a manner peculiar to itself, its pulsations must also follow peculiar laws of condensation and motion, by which he thinks the sound will be characterized.

It is agreed on all hands, that the construction of the organs of speech so far resemble a reed organ-pipe, that the sound is generated by a vibratory apparatus in the larynx, answering to the reed, by which the pitch or number of vibrations in a given time is determined; and that this sound is afterwards modified and altered in its quality, by the cavities of the mouth and nose, which answer to the pipe that organ builders attach to the reed for a similar purpose. Accordingly, the whole of the phenomena I am about to describe, will be found to result from the application of reeds to pipes and cavities of different and varying magnitude.

Now it is important to the success of these experiments, that the tone produced by the reed should be as smooth and pure as possible. The coarse tone of the common organ reed completely unfits it for the purpose, and hence we find both Kempelen and Kratzenstein endeavouring to ameliorate it. Kempelen made the tongue of ivory, and covered its under side as well as the por-

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* Schol. II. Prob. 80.

tion of the reed against which it beat, with leather. Kratzenstein succeeded better, by introducing a most important improvement in its construction. Instead of allowing the tongue to beat upon the edge of the reed, he made it exactly to fit the opening, leaving it just freedom enough to pass in and out during its vibrations*. By this construction, when carefully executed, the tone of the reed acquires altogether a new character, becoming more like the human voice than any other instrument we are acquainted with, besides possessing within certain limits the useful quality of increasing its loudness, with an increased pressure of air without altering its pitch.

All the reeds I have made use of are constructed on this principle, in one or other of the forms represented in Figs 1 and 2. They are all attached to blocks *fgh* furnished with a circular tenon *fg*, by which they can be fitted into the different pieces of apparatus represented in the plate; their place in all the figures being distinguished by the letter *R.* In Fig. 1, *ab* is a tongue of thin brass fixed firmly at its upper extremity *a*, but capable of vibrating freely in and out of a rectangular aperture in the side of the small brass tube or reed *ced*† which it very nearly fits. This is the original construction of Kratzenstein. In Fig. 2, *ab* is the tongue, attached at *a*, and capable of vibrating through an aperture, which it very nearly fits, in a brass plate screwed

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* Both these expedients are mentioned by M. Biot (*Physique* II.) but he has ascribed the first to M. Hamel (p. 170) the second to M. Grenié (p. 171) being apparently not at all aware of the existence of these Memoirs of Kratzenstein and Kempelen. There can be very little doubt but that Kratzenstein is to be regarded as the true inventor of this *anche libre*. His paper was published in 1780. See Young's *Nat. Ph.* Vol. I. p. 783.

† This brass tube is the *reed*, I believe, in the modern language of organ builders, and *ab* the *tongue;* but the term *reed* is often applied to the whole machine, and as such I have used it in this paper. In fact, in its primitive form, I suspect the whole was cut out of a single piece of reed or cane near a joint as in Fig. 3, which is copied from Barrington's drawing of the reed of the Anglesey *Pibcorn* (*Archael.* III. 33.)

to the upper surface of the block*. This is similar to the construction of the *Mundharmonicon*, or *Eolina*, lately introduced into this country from the continent, but it appears to have been originally suggested by Dr. Robison†.

My first object being to verify Kempelen's account of the vowels, I fitted one of my reeds *R* to the bottom of a funnel shaped circular cavity open at top, of which *Z*, Fig. 5, is a section (the pipe *TV* standing on the wind-chest in the usual way)‡, and by imitating his directions for the positions of the hand within the funnel, I obtained the vowels very distinctly§. I soon found

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* A section of Fig. 2, is seen in its place at *R*, Fig. 11. In all the other figures *R* is a section of Fig. 1.

† Art. Musical Trumpet. *Enc. Britt.* Supplement to 3d ed. 1801. Works IV. 538. Wheatstone in Harmonicon, Feb. and Mar. 1829.

‡ A reference to Fig. 13, in which a pipe is represented as standing on the wind-chest, will serve to illustrate the subject to persons unacquainted with the structure of the organ, and at the same time afford me an opportunity of defining certain technical terms which I shall be compelled to make use of.

A large pair of bellows kept in motion by the foot, and so constructed as to afford a constant pressure is connected with a long horizontal trunk, or *wind-chest*, of which *klno* is a cross section, and which is therefore constantly filled with condensed air. The upper side of this trunk consists of a very thick board *pkql*, which is pierced with a number of passages similar to *efg*, every one of which is furnished with a valve or *pallet kl* moving on a joint at *k*, and kept closed by a spring *m*; a wire terminating in a knob or key *r*, passes through a hole in the upper board, and rests on the pallet, so that on pressing down this knob the pallet opens and a current of air immediately rushes through the corresponding passage *efg*, and passes into any pipe *AT* which may be placed over the aperture. In this instance *R* is the vibrating reed, to be set in motion by the current. The term *portevent* is always used for that part of the tube *TR* which lies between the reed and the wind-chest, and the term *pipe* for the portion *BA*, which is between the reed and the open air, and it is to be understood that in all the figures the lower end of the tube *TV* is supposed to be placed upon one of the holes of the wind-chest. The pressure of the air is always measured by the number of inches of water it will support in a common bent tube manometer; this is generally about 3 inches in organs.

§ I think it necessary to mention that the whole of the experiments described in this paper were performed before the Philosophical Society on the same evenings that the paper was read.

that on using a shallower cavity than his, these positions became unnecessary, as a flat board *LM* sliding on the top of the funnel, and gradually enlarging the opening *KL*, would give the series U, O, A, just as well, and by making the funnel much shallower, as in Fig. 4*, I succeeded in getting the whole series in the following order: U, O, A, E, I†. Cylindrical, cubical and other shaped cavities answered as well, under certain restrictions, but I forbear to dwell on this form of the experiment because subsequent ones have rendered this more intelligible, and I merely mention it to shew the steps by which I was led from Kempelen's original experiment.

The success of this attempt induced me to try the effect of cylindrical tubes of different lengths, and for the more complete investigation of this case, I constructed the apparatus represented in Figs. 6 and 7. *TV* is a tube or ***portevent*** bent at right angles, and connected with the wind-chest at the extremity *T*, this tube terminates in a wooden piston *PQ*, provided with a socket for the reception of the reed *R* which is represented in its place. A piece of drawn telescope-tube *ABCD* is fitted to the piston which is leathered so as to be air-tight, but allows the tube to be drawn backwards or forwards, so as to alter the length of the portion *PB* beyond the reed at pleasure; the horizontal position is given to the tube to make it more manageable. There are other tubes *EFGH* of the same diameter as *ABCD* furnished with sockets at *EG* which fit on to *BD*, and their lengths are different multiples of *AB* ‡.

If therefore any reed be fitted to *PQ*, and the tube *AB* be gradually drawn out, it will shew the effect of applying to this

* Kempelen's funnel is 2 inches diameter at the mouth, and 3 inches deep from the mouth to the reed. Fig. 4, is only $\frac{1}{2}$ inch deep and same diameter.

† I use these letters throughout with the continental pronunciation.

‡ The inside diameter of *ABCD* = 1.3 in., its length = 1 ft. 6 in., and the whole length = 12 feet when all the joints are combined.

reed a cylindrical tube of any length from nothing to *AB*, then if the tube be pushed back, and a joint *EF* equal to *AB* be fitted on, a fresh drawing out of the tube will shew the effect of any length from *AB* to double *AB*, and in this way with different joints we may go on to any length we please. The results of experiments with this apparatus, I will describe in general terms,

No. 1.

I E A O U U O A E I I E A O U

a *b* *c* *d*

Let the line *abcd* represent the length *PB* of the pipe measured from *a*, and take *ab, bc, cd*, &c., respectively equal to the length of the stopped pipe in unison with the reed employed, that is, equal to half the length of the sonorous wave of the reed.

The lines in these diagrams must, in fact, be considered as measuring rods placed by the side of the tube *ABCD* with their extremity *a* opposite to the piston *P*, the letters and other indications upon them shewing the effect produced when the extremity *B* of the pipe reaches the points so marked, and the distance therefore from these points to *a* respectively being the length of the pipe producing the effects in question.

Now if the pipe be drawn out gradually, the tone of the reed, retaining its pitch, first puts on in succession the vowel qualities I E A O U; on approaching *c* the same series makes its appearance in inverse order, as represented in the diagram,

* In speaking of musical notes, I shall denote their place in the scale by the German tablature, the octave from tenor *c* to *b* on the third line of the treble is marked once thus *c'*, the next above, twice, *c''* and so on. As a standard for the pitch, I use a pitch-pipe which is made to sound by a small pair of attached bellows yielding a constant pressure of 2.5 in. The internal dimensions are .85 by .9 in., and 1 foot long. *Lumiere* .16 in. An attached scale is graduated to shew the actual length of the pipe (that is, the distance from the bottom of the pipe to the bottom of the piston) in English inches and decimals.

then in direct order again, and so on in cycles, each cycle being merely the repetition of *bd*, but the vowels becoming less distinct in each successive cycle. The distance of any given vowel from its respective center points *a*, *c*, &c. being always the same in all.

No. 2.

IEA O U   U O AEI   IEA O U   U O AEI   IEAOU

$a_{,}$ $b_{,}$ $c_{,}$ $d_{,}$ $e_{,}$

If another reed be tried whose wave $=a_{,}c_{,}$ (No. 2.) the centers of the cycles $a_{,}$, $c_{,}$, $e_{,}$, &c. will be at the distance of the sonorous wave of the new reed from each other, but the vowel distances exactly the same as before, so that generally, if the reed wave $ac = 2a$, and the length of the pipe producing any given vowel measured from $a = v$, the same vowel will always be produced by a pipe whose length $= 2na \pm v$, n being any whole number.

When the pitch of the reed is high, some of the vowels become impossible. For instance, let the wave of the reed $=ac$ (No. 3.) where $\frac{1}{2}ac$ is less than the length producing U.

No. 3.

IEA   O   U

U   O   AEI

a b c

In this case it would be found that the series would never reach higher than O; that on passing *b*, instead of coming to U, we should begin with O again, and go through the inverse series. In like manner, if still higher notes be taken for the reed, more vowels will be cut off. This is exactly the case in the human voice, female singers are unable to pronounce U and O on the higher notes of their voice. For example, the proper length of pipe for O, is that which corresponds to the note c'',

and beyond this note in singing, it will be found impossible to pronounce a distinct O.

The short and long U however are indefinite in their lengths, the short U (as in *but*) seems to be the natural vowel of the reed, and as this is but little affected by the pipe, except in loudness, between O and *b* (No. 1.) this vowel will be found to prevail through a long space, and upon approaching *b*, to change gradually into the long U, (as in *boot*), which always appears more perfect the longer the distance *ab* can be made. When reeds of high pitch are used (as in No. 3.) the vowels always become indistinct in the neighbourhood of *b*, and with bass reeds there appears to be a series of vowels like the former, on both sides the points *b*, *d*, &c. (No. 1.) but differing from the other, both by being much less distinct, and also by each vowel occurring at twice the distance from these points that it does in the other series from *a*, *c*, &c. after the manner of No. 4, where this new series is marked with an accent.

No. 4.

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  I E A O   U   A' E' I'      I' E' A'   U   O A E I
 .__________________________.________________________.
 a                          b                        c
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Cylinders of the same length give the same vowel, whatever be their diameter and figure. This may be conveniently shewn, by attaching a reed *R*, Fig. 11, to a portevent *T*, terminating in a horizontal flat plate *WX* covered with soft leather*. Tin or wooden tubes of any figure, open at both ends, and made flat at the bottom so as to fit air-tight to the leather plane, may then be applied, as in the figure, and their effect upon the reed tried. Another such leather plane, Fig. 12, may be provided, but furnished with an embouchure at *W*, like that of a common

* The diameter of my plate = 3 inches.

organ-pipe. By means of this, we can ascertain the note proper to any cavity, such as the cone *Z* placed above it in the figure. If this be transferred to the former plane its vowel will be ascertained.

As far as I have tried this, I have always found that any two cavities yielding the identical note when applied to Fig. 12, will impart the same vowel quality to a given reed at Fig. 11, or indeed to any reed, provided the note of the reed be flatter than that of the cavity, according to the principle explained in No. 3*.

The vowel distances in the first series, that is, those measured from *a* (No. 1.) are always rather less than those measured from the center points *c*, *e*, &c. This diminution varies with different reeds, and appears to be due to some disturbing effect of the reed itself, or the short pipe annexed to it which I have not been enabled as yet to examine so satisfactorily as I could wish. For this reason I have preferred in the following table obtaining the vowel lengths from the second and third series, by bisecting their respective distances from each other measured across *c*, which appear liable to no such alterations. These lengths, in inches, occupy the third column. For want of a definite notation, I have given in the second column the English word containing the vowel in question. The fourth contains the actual note of the musical scale corresponding to a stopped pipe of the vowel length, supposing O to yield c'' which it does as nearly as possible. In effect its length $= 4.7$ inches which with Bernouilli's correction† gives 4 inches for the length of the pitch pipe, and this will be found to give c''.

* We can now connect the results of Figs. 4 and 5 with those of Fig. 6. If the cavity *Z*, Fig. 4, be placed on Fig. 12, and the flat board *EM* slid over its mouth, a scale of notes will be heard. If now any position of *LM* be taken by which *Z* is made to yield the same note as a given cylinder, then both will yield the identical vowel upon Fig. 11.

† *Mem. Ac. Par.* 1762, p. 460. Biot, *Phys.* II. p. 134.

Table I.

I	See	.38 ?	g^{v}
E	Pet	.6	c^{v}
	Pay	1	d^{iv}
A	Paa	1.8	f'''
	Part	2.2	$d''b$
A°	Paw	3.05	g''
	Nought	3.8	$e''b$
O	No	4.7	c''
U	But	Indefinite	
	Boot		

I have found this table as correct a general standard as I could well expect; for vowels, it must be considered, are not definite sounds, like the different harmonics of a note, but on the contrary glide into each other by almost imperceptible gradations, so that it becomes extremely difficult to find the exact length of pipe belonging to each, confused as we are by the difference of quality between the artificial and natural vowels. Future experiments, in more able hands than mine will, I trust, determine this matter with greater accuracy, and I should not even despair of their eventually furnishing philologists with a correct measure for the shades of difference in the pronunciation of the vowels by different nations.

A few theoretical considerations will shew that some such effects as we have seen, might perhaps have been expected. According to Euler*, if a single pulsation be excited at the bottom of a tube closed at one end, it will travel to the mouth of this tube with the velocity of sound. Here an echo of the pulsation

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* Prob. 77, and Cors. Schol. to Prop. 76. and Schol. 2 and 3, Prop. 78. in *Nov. Comm. Petrop.* XVI. *Mem. Acad. Berl.* 1767. See also *Enc. Metrop.* Art. Sound, p. 776.

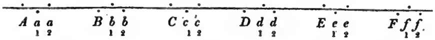
will be formed which will run back again, be reflected from the bottom of the tube, and again present itself at the mouth where a new echo will be produced, and so on in succession till the motion is destroyed by friction and imperfect reflexion. If it be a condensed pulsation that is echoed from the open end of a tube, the echo will be a rarefied one and *vice versâ*, but the direction of the velocity of translation of its particles will be the same. On the other hand when the reflexion takes place from the stopped end, the pulsation retains its density, but changes its velocity of translation. The effect therefore will be the propagation from the mouth of the tube of a succession of equidistant pulsations alternately condensed and rarefied, at intervals corresponding to the time required for the pulse to travel down the tube and back again; that is to say, a short burst of the musical note corresponding to a stopped pipe of the length in question, will be produced.

Let us now endeavour to apply this result of Euler's to the case before us, of a vibrating reed, applied to a pipe of any length, and examine the nature of the series of pulsations that ought to be produced by such a system upon this theory.

The vibrating tongue of the reed will generate a series of pulsations of equal force, at equal intervals of time, but alternately condensed and rarefied, which we may call the primary pulsations; on the other hand each of these will be followed by a series of secondary pulsations of decreasing strength, but also at equal intervals from their respective primaries, the interval between them being, as we have seen, regulated by the length of the attached pipe. Take an indefinite line Af_2 (No. 5) to represent the time, and let ABC...&c. be the primary pulsations, a_1a_2...b_1b_2... their respective secondaries, and for simplicity we will suppose that after the third they become insensible, also we will denote a condensed pulse by a stroke above the line, and a rarefied one

by a stroke below. Suppose in the first place that the intervals of the secondary pulses are less than those of the primaries, and take AB, BC, &c. $(=a)$ to represent the primary interval, and Aa_1, a_1a_2, &c. $(=s)$ the secondary. The order of the pulses in this case will be evidently that represented in the diagram.

No. 5.



Suppose now that the secondary interval instead of being $=s$, should $=2a+s$,

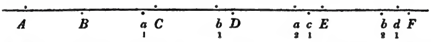
No. 6.

then taking ABC... at equal intervals $(=a)$ as before, to represent the primary pulses, we must place a_1 at the interval $2a+s$ from A, that is, at the distance s from C, a_2 at the distance $2s$ from E, and so on, and similarly for $b_1 b_2$, &c. but in this way we plainly get, after the four first, a series of pulses precisely similar to those we obtained before, both in order of intervals, succession of intensity, and alternations of condensed and rarefied pulses. If we take the secondary interval $=4a+s$, we shall find the same series occurring after the eighth primary, and in like manner when the secondary interval $=2na+s$, we always obtain the same succession of pulses after the $4n$ first which may be neglected, and as we may take the distances AB, Aa_1, &c. to represent the length of stopped pipes, which give musical notes, the interval of whose pulses are in that ratio respectively; we may say that whatever effect be produced on the ear by applying a pipe, length $=s$ to a vibrating reed whose note is in unison with a stopped

pipe, length $= a$, will also be produced by applying to the same, a pipe, length $= 2na + s$, n being any whole number.

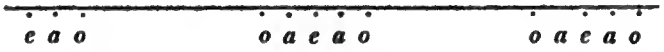
Let now the secondary interval $= 2a - s$, a similar process will give the following series.

No. 7.



This after the four first is exactly similar to the former in the order of the intervals of the pulses, and the alternation of condensation and rarefaction, and only differs from it by the little groups of pulses increasing in intensity in this case, and diminishing in the other, a difference scarcely worth noticing, so that we may now say, that whatever effect be produced by applying a pipe $(= s)$ to reed $(= a)$, the same will be produced by a pipe $2na \pm s$.

No. 8.



Hence if a given effect a be produced by a pipe whose length is ea (No. 8.) the same will be produced by a pipe whose length is $ee - ea$, $ee + ea$, &c. (taking $ee = 2a$). Again, if another effect o be produced by a pipe $= eo$, the same will be produced by a pipe $= ee - eo$, $ee + eo$, &c. and so on, and in this way it appears that if upon gradually lengthening the pipe we find a certain series of effects produced at the beginning, we shall have the same series in inverse and direct order alternately, the centers of each set of effects being separated by an interval $(= ee)$ the length of the open pipe in unison with the reed. So far we find a perfect agreement with experiment, and it is plainly impossible to extend our reasoning much further with respect to the effect which

a series of pulsations of this kind might be expected to produce on our organs of hearing.

If we examine the nature of our series, we shall find it merely to consist of the repetition of one musical note in such rapid succession as to produce another. It has been long established, however, that any *noise* whatever, repeated in such rapid succession at equidistant intervals, as to make its individual impulses insensible, will produce a musical note. For instance, let the musical note of the pipe be g'', and that of the reed c', which is 512 beats in a second, then their combined effect is g''...g''...g''...g''...(512 in a second) in such rapid equidistant succession as to produce c', g'' in this case producing the same effect as any other noise, so that we might expect *à priori*, that one idea suggested by this compound sound would be the musical note c'.

Experiment shows us that the series of effects produced are characterized and distinguished from each other by that quality we call the vowel, and it shews us more, it shews us not only that the pitch of the sound produced is always that of the reed or primary pulse, but that the vowel produced is always identical for the same value of s. Thus, in the example just adduced, g'' is peculiar to the vowel A° (Tab. I.): when this is repeated 512 times in a second, the pitch of the sound is c', and the vowel is A°: if by means of another reed applied to the same pipe it were repeated 340 times in a second, the pitch would be f, but the vowel still A°. Hence it would appear that the ear in losing the consciousness of the pitch of s, is yet able to identify it by this *vowel* quality. But this vowel quality may be detected to a certain degree in simple musical sounds; the high squeaking notes of the organ or violin, speak plainly I, the deep bass notes U, and in running rapidly backwards and forwards through the intermediate notes, we seem to hear the series U, O, A, E, I,

I, E, A, O, U, &c. so that it would appear as if in simple sounds, that each vowel was inseparable from a peculiar pitch*, and that in the compound system of pulses, although its pitch be lost, its vowel quality is strengthened. To confirm this, we ought to be able to shew that the note peculiar to each vowel in simple sounds, is identical with that of the secondary pulse in the compound sound, that is, with the note produced by a stopped pipe of the same length as that in Tab. I. corresponding to the given vowel, and as far as I have tried, it appears to be the case, but there is so much room for the exercise of fancy in this point, from the difficulty of fixing on the exact vowel belonging to a simple sound, that I do not mean to insist upon it.

No. 9.

$$\dot{A} \qquad \dot{B}\,\underset{1}{\dot{a}} \qquad \dot{C}\,\underset{1}{\dot{b}}\,\underset{2}{\dot{a}} \qquad \dot{D}\,\underset{1}{\dot{c}}\,\underset{2}{\dot{b}} \qquad \dot{E}\,\underset{1}{\dot{d}}\,\underset{2}{\dot{c}}$$

No. 10.

$$\dot{A} \qquad \underset{1}{\dot{a}}\,\dot{B} \qquad \underset{2}{\dot{a}}\,\underset{1}{\dot{b}}\,\dot{C} \qquad \underset{2}{\dot{b}}\,\underset{1}{\dot{c}}\,\dot{D} \qquad \underset{2}{\dot{c}}\,\underset{1}{\dot{d}}\,\dot{E}$$

If we were to take the secondary interval $= a + s$ or $a - s$, we should obtain series of pulses like those in Nos. 9 and 10. These being after the two first precisely similar to No. 5, in the order of intervals and succession of intensity, would seem, at first sight, to prove that we ought to have a direct and inverse series of vowels like the others, on both sides of *b*, No. 1,

* Kempelen has a curious remark about this, §. 110....... "Il me semble, que lorsque "je prononce des voyelles differentes sur le même ton, elles ont pourtant quelque chose "qui donne le change à mon oreille, et me fait penser qu'il y a une certaine melodie, "qui cependant, comme je le sais très-bien ne peut-être produite que par la variation "des tons en aigus et graves......"

and similarly, if the secondary interval were taken $= a \pm \overline{2n+1}.s$, we ought to have the same on both sides of *d, f,* &c. The pulsations in each of these groups, however, are either all condensed or all rarefied. Near *b*, therefore, they will tend to coalesce, and will scarcely be sufficiently distinguished from each other to impart the vowel quality in the same way that groups of alternate pulsations do. When they suggest any musical note at all, it will plainly be an octave higher than if the pulsations were alternate, because in the latter case, the interval between one *condensed* pulse and another, is twice that in the former. As the vowels have been shewn to be identified by the musical note of the secondary groups, it follows, therefore, that each vowel will, in this new series, be at twice the distance from *b, d,* &c. that it is in the original series from *a, c, e,* &c. I have already mentioned, that vowels of this kind are to be found on both sides of these points, *b, d,* &c. (See No. 4.)

Having shewn the probability that a given vowel is merely the rapid repetition of its peculiar note, it should follow that if we can produce this rapid repetition in any other way, we may expect to hear vowels. Robison and others had shewn that a quill held against a revolving toothed wheel, would produce a musical note by the rapid equidistant repetition of the snaps of the quill upon the teeth. For the quill I substituted a piece of watch-spring pressed lightly against the teeth of the wheel, so that each snap became the musical note of the spring. The spring being at the same time grasped in a pair of pincers, so as to admit of any alteration in length of the vibrating portion. This system evidently produces a compound sound similar to that of the pipe and reed, and an alteration in the length of the spring ought therefore to produce the same effect as that of the pipe. In effect the sound produced retains the same pitch as long as the wheel revolves uniformly, but puts on in succes-

sion all the vowel qualities, as the effective length of the spring is altered, and that with considerable distinctness, when due allowance is made for the harsh and disagreeable quality of the sound itself.

But there is another remarkable phenomenon to be observed in the experiment with Fig. 6, which, as it is wholly unconnected with the vowels, I have hitherto, to prevent confusion, omitted to notice. I stated that during the experiment the reed retained its pitch, in fact however there is a considerable change in it which I proceed to describe.

In Fig. 18, the indefinite line *abcb'*... is taken, as before, to represent the measuring rod; the ordinates of the curve, traced below this line as an axis, are the lengths of the pitch-pipe in unison with the note produced by the apparatus, when the extremity of the pipe reaches their respective feet*. Let *ab*, *bc*, *cb'*... be respectively equal to the half-wave of the reed, therefore *c* is the center of the second series of vowels.

Let the pipe be gradually drawn out; the pitch will remain constant till its extremity approaches *f*, about midway between *a* and *b*. Here the note begins to flatten (which is represented by the increase of the ordinates), proceeding still farther, it continues to flatten till the pipe reaches a point *n* beyond *b*, where the note suddenly leaps back to one about a quarter tone sharper than the original, which however it soon after drops gradually to, and preserves till the pipe reaches *c*; proceeding from *c* through the successive cycles, the same phenomena will be found at the

* To save room on the plate, but at the same time to shew the real nature of this change of pitch, these ordinates were constructed for the particular case of a reed (pitch =6.6) and drawn to the same scale as the rod ($\frac{1}{4}$ of the original) but taken equal, not to the length of the pitch-pipe, but to the excess of its length above six inches. If the line *ab* were drawn an inch and a half higher up, the ordinates would be exactly proportional to the real ones. (See Note A.)

corresponding points of each, but not in so great a degree. The whole amount by which the note is flattened varies, but is generally about a tone, it is accurately stated in the Tables*.

The point n is by no means fixed, a jerk of the apparatus, a too hasty motion of the pipe, will make the reed leap back to the original note much sooner than it would do with a very gentle and gradual motion; it varies too with different reeds, although their pitch be the same, and it varies with the pressure of the bellows, the note changing sooner with a greater pressure.

Now if we begin from any point beyond n, and shorten the pipe gradually, its extremity may be brought considerably behind n, (say to m) before it leaps back to the other note, so that in fact at any point p between m and n there are two notes pq, pr, which it may be made to produce, neither of them exactly its proper one, but one a little *flatter* and the other a little *sharper*. Some reeds upon fixing the extremity of the pipe between m and n, may be made to change these notes from one to the other by a dexterous jerk of the bellows upwards or downwards, and to retain either of them for any length of time, at pleasure.

With others again there is a point between m and n, where the two notes appear actually to be heard at once, producing a most singular effect. The real fact in this case seems to be that the reed is producing its two notes alternately, but changing quickly and periodically from one to the other, so that they seem to be both going on at once.

To produce these effects completely, a considerable pressure of wind is sometimes required†. Should the reed be stiff, or the pressure on it not sufficiently great, it becomes quite silent for a considerable extent on both sides of p.

* Vide Note A. † As much as six inches.

The note always increases in loudness on approaching *f*, and diminishes when it begins to flatten. After the leap it is commonly husky and bad for a little distance, and at every point *c*, that is, whenever the pipe is a multiple of the whole wave of the reed, the tone of the reed appears to be perfectly unaffected by the pipe, and is, if any thing, rather less loud than it would be without it.

Similar phenomena have been imperfectly observed by the old organ builders, and have been described by Robison, who set a reed in a glass foot, and adapted a sliding telescope tube to it. Again by Biot, who also used a glass foot, but made the length of the reed to vary instead of that of the pipe*. All these experiments were made however with the old-fashioned reed, whose oscillations were disturbed by the tongue beating on the edge of the tube, so that the phenomena could not be defined with so much precision as they are, when the free reeds are made use of. To discover how the motion of my reed was affected, I therefore adapted a telescope tube *ABC*, Fig. 17, to a glass *portevent DdEeT*. Behind the tongue *R*, but at such a distance as not to disturb its action, I placed a micrometer scale *mn*, supported by a wire *o*, by which I could measure the extent of its excursions towards *m*, which are very well defined: of course in the other direction they are concealed by the brass reed. I found that the excursions were constant as long as the pitch remained constant: when the flattening or sharpening took place, the excursions were diminished. (See Note B.) The ordinates of the dotted curve above the axis, in Fig. 18. represent the semi-excursions of the reed in this experiment. It was remarked that when the reed was managed so as to produce the double

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* *L'Art du Facteur d'Orgues*, par D. B. de Celles, pp. 439, 440. (Vide Note E.) Robison, *Works* IV. 508, and *Enc. Brit.* Biot, *Physique* II. 169.

note, that its excursions were no longer well defined, but it seemed thrown into strange convulsions.

Let us now consider how to account for these effects. We have seen that an aerial wave travelling along a pipe will be regularly deflected backwards from the extremity, changing the sign of its density, and preserving that of its velocity when the pipe is open; but if closed, then retaining the sign of its density, and changing that of its velocity. Let the curve (Fig. 19.) represent in the usual manner, the densities of the series of waves generated from a reed R, in a pipe AB, and proceeding in the direction of its axis, the ordinates above the axis indicating condensation, and those below, rarefaction.

To take the simplest case, let the pipe be stopped at A and B. The series therefore will have been reflected from B to A, and back again continually, diminishing in force each time, till the effect of the waves becomes insensible. The whole effect upon any given particle within the pipe at any time, will, upon the principle of the superposition of small motions, be the sum of the effects produced by all these reflected waves, upon that particle at that moment. If therefore BA', $A'B'$, &c. be taken equal to AB, the actual density of every point of the pipe, at the instant when any given portion R of a wave is issuing from A, will be represented by the curve formed by combining all the alternate portions AB, $A'B'$ taken directly, with all the portions BA, $B'A''$,...&c. taken reversely, and allowing for the gradual diminution of force.

If B be open, the curve must at each point B, B', &c. be reversed with respect to the axis, as is shewn by the dotted lines, since the density is reversed at the open end of the pipe.

Suppose now that the tube is exactly the length of a wave, as AB, (Fig. 20,) and first let A and B be both closed. In this

case the direct portions A_1B_1, A_2B_2, &c. will be exactly similar to AB, also all the retrograde portions A_1B, A_2B_1 will be exactly similar whatever portion of the wave is issuing from A, therefore all the direct waves will unite and assist each other, and in like manner all the retrograde waves, so that the column of air in the pipe will vibrate in the well known manner of a common organ-pipe. We should obtain a similar result if the pipe had been taken equal to any multiple of the wave.

But if B be open, the alternate portions BB_1, B_2B_3, &c. will be inverted: in this case A_1B_1 being exactly similar to AB, but on the opposite side of the axis, will tend to counteract it; again A_2B_2 destroys A_1B_1 and so on, and in like manner the retrograde waves alternately tend to neutralize each other, so that, in fact, were it not for the gradual decay of the secondary pulsations, there ought to be no sound at all in this case.

Again, if we consider a tube ab equal in length to half a wave, we shall find, in a similar way, that when b is stopped, the secondary pulses mutually *interfere* and destroy each other, and when open that they strengthen each other, and the same results would be found if the length of the pipe were taken equal to any odd multiple of the half wave. In the same way we may examine the case of pipes open at both ends, and so generally we find, that in pipes stopped or open at both ends, the pulsations strengthen each other when the pipe equals a wave or its multiple, and destroy each other when the pipe equals a half wave or its odd multiple, and that in pipes stopped at one end, the reverse is the case.

This mutual destruction of the pulsations, when the pipe is a multiple of the wave is confirmed by experiment. It has been shewn that such a pipe rather weakens than strengthens the tone of the reed.

We may now be partly able to see how the reed is affected

by different lengths of pipe. Its motion, taken alone, is well known to be considerably under the dominion of the current of air by which its pulsations are maintained*, and if a pipe be attached to it, it will plainly be also subjected to the periodic return of the secondary pulsations, by which we may expect to find the time and mode of its oscillations affected. We have seen, however, that when the pipe is about the length of some odd multiple of the half wave, the secondary pulses are united. Here it is then that we may look for the greatest disturbance of the motion, and accordingly our experiments have shewn us, that the pitch of the reed, and consequently its oscillations, are only affected on approaching such lengths of the attached pipe. At other lengths it would appear that their impulses being separated, and falling in succession on the reed, are incapable of producing any sensible effect on its motion, although they are able to qualify the tone by imparting to it the vowel qualities.

Now as the reed moves outwards, a rarefied pulse is generated, which travels to the extremity of the pipe, and returns condensed; and supposing the pipe equal half a wave, the beginning of this secondary pulse will just coincide with the extremity of the first, so that as the reed returns, it is met by this reflected pulse, which tends to check the extent of its excursion, and, as it seems, to increase the time. The condensed pulse now generated will, in like manner, present a new secondary rarefied one, which will retard the reed during its outward motion, and so on.

But as the length of the pipe was taken equal to half the wave of the reed in its *free* oscillations, it appears that the wave now produced, since the note is flatter, will be of greater extent,

* Biot, *Physique* II. 167.

so that the pipe may be lengthened to correspond to the new value of the half wave. But this being done, the same reasoning will shew that the excursion of the reed should be still further diminished in extent, and augmented in time; and in this way we may go on increasing the length of the pipe, diminishing the extent of the excursions, and flattening the note, till we reach some point where it will be easier for the reed to oscillate in the original mode, and there it will of course return to that mode of vibration. But the length of pipe at which this change takes place, will manifestly depend upon the elasticity of the reed, and the pressure of the current, and may therefore vary with different reeds and pressures as we have found it to do.

But we have seen that when the interval of the secondary pulses is somewhat greater than that of an oscillation of the reed, there is a slight tendency to accelerate the motion in diminishing its extent. If this once takes place, a similar line of reasoning will shew, that upon shortening the pipe to accommodate it to the new wave, it must go on rising in pitch till it reaches some point where it will be easier for it to vibrate in the other way.

That the lengths of pipe producing all these alterations of pitch are unfavorable to the reed's motion, is proved by the fact, that with them a less weight of wind reduces the reed to silence, although such less weight is quite sufficient to make the reed speak freely with favorable lengths. It also appears by the same test, that the lengths with which the sharpening takes place are considerably more unfavorable to the reed's motion, than those which flatten it.

The fact of certain lengths of pipe being unfavorable to the motion of the reed, naturally recalls the experiments of M. Greniè*. This gentleman found, that with some notes of the scale, he

* Biot, *Physique*, II. 173.

was obliged to alter the length of the *portevent*, or tube conveying the wind from the bellows to the reed, as there appeared to be some given length of *portevent* for each reed, which completely prevented the reed from speaking, but it did not appear that these lengths followed any law. As this phenomena appeared to be of the same nature, as that I have just described, I set about to investigate it with the following apparatus.

By way of *portevent* I took two brass telescope tubes *AB*, *CD*, Fig. 16, sliding tightly over each other, and each a foot in length, the internal diameter of *CD* being .5 inches, at *A* was a socket for the reception of the reed *R*. Two additional tubes (as *EF*), with sockets, served to increase the length, so that by combining the use of these with the slide *BC*, I could try the effect of a *portevent* of any length from one foot to four.

For shorter lengths than this, I made use of a pipe *AB*, Fig. 13, to the end of which was attached a piston *BV*, carrying a reed *R*, and fitting air-tight in the *portevent* *TVC*, which is represented as standing on the wind-chest of the organ.

The note of the first reed I tried was = 4.7 inches by the pitch-pipe, and upon drawing the piston *BV* up by means of its attached pipe *AB*, the reed refused to speak, when $TV = 3.4$ in. and sounded again when $TV = 5.5$

Now according to the former experiments, this should have begun to happen when *TV* was nearly equal to the half wave; but the note of the reed I employed = 4.7 inches by the pitch-pipe, which, corrected by Bernouilli's Table, gives about 5.3 for the half wave, so that the *portevent* was much too short for my theory*. The cause of this anomaly appeared to lie in the short passage *efg* already described, which conveys the wind from the

* Especially as the waves in these experiments always come out rather *longer* than the corrected length of the pitch-pipe, (See Note A.)

wind-chest to the *portevent*. To get rid of this, I fairly set the end of the *portevent* into the lid of the bellows, as represented in Fig. 14, so that the lower extremity T of the pipe, or vibrating column, should be perfectly well defined. Now indeed the reed became silent, when $T'V' = 5.9$, and spoke again at about 8, leaving a difference in length from the former experiment of 2.5 inches, so that there can be little doubt, but that the passage *efg* is to be reckoned as part of the *portevent*, and perhaps may serve to account for the anomalous results obtained by M. Grenié. Pursuing this enquiry, by means of the longer tubes, Figs. 15, and 16, I found as I expected, that these intervals of silence, flattening, &c. occurred regularly, whenever the *portevent* was made nearly equal to some odd multiple of the half-wave of the reed*. There are faint indications of vowels in this case. (See Note C).

I shall conclude with a few experiments, which appear to confirm the views already laid down. Take a piston MN, Fig. 8, very nearly of the diameter of the tube $ABCD$, Fig. 6.† By means of the slender handle o, it may be made to slide up and down the tube, as represented in Fig. 8. If this is done, the sound of the reed goes on without interruption, putting on, but rather less distinctly than before, the vowel qualities, in direct and inverse order alternately, corresponding to the length of tube BN, between the end of the piston and the mouth, whatever be the length of PB: while, on the other hand, whenever the length PM is made nearly equal to the whole wave of the reed, or some multiple of it, the flattening and other phenomena take place,

* As therefore there are certain lengths of *portevent*, which prevent the reed from speaking, it is necessary to have the power of adjusting TVR, Fig. 6, to the reed made use of: this is done by a sliding joint at V, which may be fixed to the required length by a clamp, all this is omitted in the figure to avoid confusion.

† Diameter of tube = 1.3 in., of piston = 1.25 in.

which in the former experiments were found to occur only at odd multiples of the half length. In this case, one portion of the pulses generated by the reed, is propelled past the sides of the piston to the mouth of the pipe, and thence echoed backwards and forwards between *B* and *N*, producing the vowels, and the remaining portion echoed between *M* and *P*, as if *PM* were a pipe stopped at both ends, in which case we have seen that the secondary pulses unite, and therefore disturb the motion of the reed, when the pipe equals the whole wave, or a multiple of it.

If the reed *R*, instead of being inserted into the piston as in Fig. 6, be fixed to the end of a slender tube *WR*, (Fig. 9), which slides air-tight through a collar of leather attached to the piston, so that the length *PS* admits of alteration at pleasure, the vowels and other phenomena are still found to depend upon the length *PB*, (all those of course that would have been produced from *P* to *S* being lost.)

If the reed be attached to the end of a *portevent TVR*, (Fig. 10), and presented to the tube *ABCD*, which is provided with a solid piston *PQ*, the vowels and other phenomena are still produced as before, and still depend upon the actual length *PB* of the greater tube.

There is one curious circumstance about the two last experiments, that although the flattening takes place only when *PB*, (Figs. 9, and 10.) is equal to the half wave of the reed, or some odd multiple of it, the amount of it varies with the distance of the reed from the mouth of the tube; being greatest, when *BS* is the half wave, or an odd multiple of it, and almost imperceptible, when *BS* is the whole wave, or a multiple, or when *BS* vanishes, that is, when the reed is exactly at the mouth of the pipe*. This appears to depend upon the different variation

* See Note (D).

of density, at different parts of the pipe. We have seen, that when the pipe was equal to the half wave, or its odd multiples, that the variation of density took place as in a common organ-pipe, being nothing at the mouth, or at a multiple of the wave from it, where our flattening nearly disappears, and greatest at the distance of the half wave, or its odd multiples, where our flattening is greatest. Now as it has been shewn, that this alteration of pitch is occasioned by the reaction of the secondary pulses which disturb the motion of the reed by a periodic variation of density, it is clear, that if in the stratum of air *S*, (Figs. 9, and 10,) where the reed is placed, this variation is destroyed or diminished by the pulses returning from *BD*, that the effect on the reed's motion will be proportionally diminished.

Lastly, if instead of presenting a stopped pipe to a reed, as in Fig. 10, we substitute one open at both ends, such as a telescope without the glasses, or for shorter lengths, tin tubes, the flattening takes place when the tube equals the wave, or a multiple of it, and the vowels at double the length they did with stopped pipes. That this ought to happen is plain from what has been already said, and from the explanation of Nos. 9, and 10, which it will be easily seen, are the vowel diagrams for this case.

Some useful hints may now be deduced, for improving the construction of the reed-pipes of organs. Organ builders have been in the habit of attaching pipes to reeds, for two purposes. First, as in the ordinary reed stops, by applying to the reed a conical pipe, giving the same note, they hoped that the vibrations of reed and pipe would assist each other, and the whole effect be improved, and this view of the matter has always been taken by the theorists: the builders however have been perplexed by finding the motion of the reed obstructed, and its note flattened by such a pipe, and have therefore always made the pipe

a little sharper than the reed, and its principal use appears to be, that by its conical shape, it augments the loudness of the note, on the principle of the speaking trumpet. But we have seen, that although when the reed and pipe give the same note, the undulations of the pipe disturb the motions of the reed, whether the pipe be open or stopped at both ends, or stopped at one end only; (always recollecting that the organ builder's reed-pipe must be considered as stopped at the reed end), yet that the amount of this disturbance varies with the place of the reed in such a pipe, being greatest at the bottom of the pipe, and least at its mouth*, so that it seems that the place hitherto chosen for the reed is the worst possible, and that it ought rather to be at the mouth of the pipe, in short that the best arrangement would be that of Fig. 10, supposing the reed to be at *BD*, and the pipe placed vertically. This too would give the means of conveniently tuning the pipe to the reed by the plug *PQ*. With this arrangement the note of the reed is greatly augmented, and its motions being undisturbed, it speaks more readily.

The other application of pipes to reeds has been in the vox-humana stop, where very short pipes, not in unison with the reed, are employed: the principles of this, however, have never been thoroughly understood, and the whole has always been a puzzle to the builders†.

If by a voxhumana stop, we understand a stop possessing the same vowel quality in every note, the thing is easily to be

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* See Note (D).

† According to Biot, the pipes of the voxhumana are nearly of the same size, (*Phys.* II. 171.) but in the diapason given by B. de Celles, they differ considerably, although very short (*Facteur d'Orgues*, pp. 84, 366). Their form is a cylinder set upon a truncated inverted cone, and half closed at the mouth. The length of $cc = 9.1$ in. diameter 1.4 in. length of $c''' = 3.3$ in. diameter 1 in.

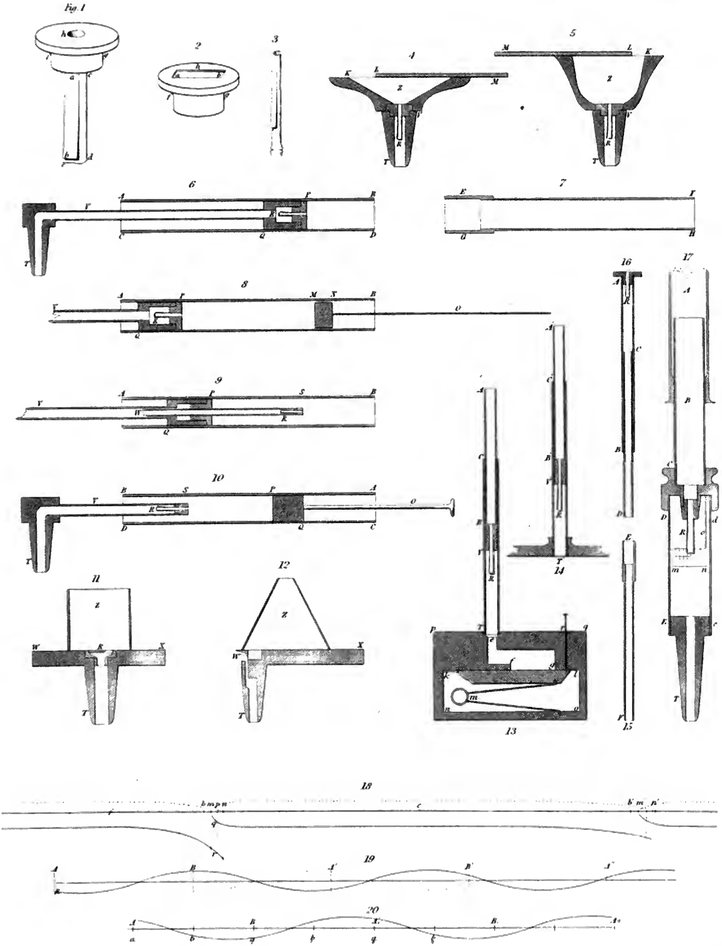
obtained on the principles I have laid down, and I think would be a source of very pleasing variety; but it must be remembered, that there will always be a natural limit to the extent of such a stop upwards, from the impossibility of imparting each vowel quality to the notes of the scale beyond its own peculiar one. Thus we might have an O stop, but it could not extend above c'', an A stop might reach to f''', and an E to d^{v} *, and so on.

I shall here, for the present, conclude; reserving the application to the action of the human organs, of the facts and principles I have endeavoured to bring forward, to a future opportunity.

* The pipes attached to the reeds might be made to give E, and shades be brought over all their mouths simultaneously, by a pedal, so as to enable them to give any vowel at pleasure, imitating Fig. 4, where however the cavity *Z* may be made of any convenient form, and the shade *LM* need not rest on its edges, but be made to approach the mouth in any way most suitable to the mechanism employed. The common reeds *may* be used, but their vowels are by no means so distinct as those produced by the free reeds, which should always be employed in experimenting.

ROBERT WILLIS.

CAIUS COLLEGE,
March 16, 1829.



J W Lowry, Sculp

APPENDIX.

Note A.

The following Table will serve to illustrate the remarks on the change of pitch, and, in fact, is the one from which Fig. 18. was constructed. The pitch of the reed = 6.6 inches or g' nearly. I have in all the Tables denoted the pitch by the actual length of the pitch-pipe, as the most accurate method; all the measures are in inches and decimals.

Table II.

Length of tube *PB*, Fig. 6.		Pitch.		
	4	6.63		
	5	6.71		
	6	6.76		
	7	6.85		
	8	7.1		
1st duplication.....	8.74 = *am*			6.48
	8.8 = *ap*	7.5	or	6.45
	8.89 = *an*	7.55		
	9			6.53
	10			6.6
center of 2nd Cycle	17.1 = *ac*			
2nd duplication.....	26 = *am'*	7.03	or	6.5
	26.45 = *an'*	7.11	—	6.53
center of 3rd Cycle	34.7			
4th duplication......	43.3	6.96	—	6.53
	44.1	7.01	—	6.58
5th duplication......	60.7	6.87	—	6.55
	62	6.96	—	6.6

Table III.

Pitch of Reed = 2.5.							
Length of tube at successive doubling points as the pipe was lengthened.	4.75	13.8	23.1	32.15	41.45	50.85	59.9

Table IV.

Pitch of reed, an octave above 4.2. In this case the pitch of the reed being high, the vowels were all lost. The reed, instead of altering its pitch, and doubling at successive periods, became silent for a little space and then resumed its note, but always gradually increasing in loudness to a painful degree, just before its silent places.

Length at silent points.	2.55	8.9	15.3	21.7	28.05	34.45	40.85
—— when note resumed.	3.65	9.9	16.1	22.55	28.9	35.2	41.6

I have stated that the cycles of phenomena recur at distances equal to the sonorous wave of the reed. I must remark, however, that this distance is always somewhat greater than it ought to be, if calculated from the length of the pitch-pipe corrected by Bernouillis's Table. (Biot. *Phys.* II. 135. *Mem. Acad. Par.* 1762.) The difference is about an eighth or tenth of the wave. Thus,

Reed, Tab. II. by a mean of the different distances gives the half wave.......... = 8.8
But pitch = 6.6, and this corrected......... = 7.2

Diff. = 1.6

Reed, Tab. III. gives half wave............ = 4.65
Pitch = 2.5, and corrected.................. = 3.4

Diff. = 1.25

Reed, Tab. IV...................... half wave = 3.2
Pitch ...... = octave above 4.2
Corrected = ... 4.95, ∴ true pitch = 2.475

Diff. = .725

NOTE B.

Table V. contains a set of experiments with Fig. 17, and may serve to shew the effect which the length of the pipe, and the force of the wind has on the excursions of the reed. The reed employed, was one of those which produced the double sound.

TABLE V.

Pressure 6 inches.			Pressure 4 inches.			Pressure 2 inches.		
Length.	Pitch.	½ Excursion.	Length.	Pitch.	½ Excursion.	Length.	Pitch.	½ Excursion.
12	11.05	.076	12	10.93	.07	12	11	.046
15	11.52	.06	14	11.13	.066	14	11.1	.03
16	11.6	.05	15	11.3	.056	15	11.3	.03
17		.046	16	11.6	.046		Silence	
18		.036	17	11.8	.034	17	9.6	
18.75		.03	18	Silence		18	10.2	
19	Double	.024?	19	10.65?	.02	20	10.4	.02
19.5		.03	19.4	10.7	.04	21	10.75	.03
20	10.8	.06	20		.052	24 to 33	10.87	.054
21	10.87	.07	21	10.75	.064			
24 to 33	10.9	.08	22 to 33	10.8	.074	40.8	11.2	.030
35	11.05	.076	34	10.83	.074			
39	11.15	.07	35	10.95	.072			
40	11.24	.068	39	11.1	.066			
40.8	11.4	.06	40.8	11.3	.05			

Note C.

Table VI. is a set of results from Fig. 16, applied to the wind-chest, as in Fig. 13. Table VII. is a set from Fig. 13, with the same reed. It will be seen, that in this case the length of *AB* affects the results as might have been expected.

Table VI.

6 Inches pressure.

Length of *DA* from Wind-chest.	Pitch.
15.25	5.2
19	5.25
20.75	5.5
22.5	5.7
23.25	5.9
23.4	Leap.
23.9	Fluttering Tone.
24.1	5.1 } Husky Tone.
24.5	
25	Good.
27.4	5.15
29.8	5.2
32	5.25
34	5.35
35.5	5.4
36.75	5.65
37.6	5.7 and Leap
38.1	5.
38.75	5.1
39.8	5.15
43.5	5.2
49.2	5.35
51.25	5.45
53.5	5.65

Table VII.

	Pitch.		
Length of *TV*.	$AB=12$.	$AB=20$.	$AB=22$.
3	5.23		
4	5.25		5.45
5	5.3		5.5
6	5.6	5.63	5.65
6.4	5.65	4.75	5.05
	Silence at 6.5		
8	4.75	5.2	5.83
	Tone bad.		
9.2	5.	5.3	5.4
10.7	5.2	5.3	5.45

Note D.

Table VIII. contains a set of experiments with Fig. 10. The tube *ABCD* was fixed with respect to *TVR* in such a manner that *R* was protruded into the tube at distances expressed in the first column. The piston was then moved till the flattening took place, and the note jumped back to the original one, or rather as before-mentioned to one a little sharper. The distance of the piston from the end of the tube, when this took place is seen in the second column. The note to which the pitch had sunk in the third column, and the sharp note to which it returned in the fourth.

Table VIII.

PITCH OF REED, 4.87.

First Leap.

Length of *BS*.	Length of *BP*.	Pitch. Flat.	Pitch. Sharp.
0	6.45	4.95	
.5	6.38	5.05	
1	6.28	5.1	4.83
2	6.03	5.15	4.65
4	6.06	5.17	4.5
5	6.19	5.23	4.4

Second Leap.

Length of *BS*.	Length of *BP*.	Pitch. Flat.	Pitch. Sharp.
0	20.78	4.95	
1	20.7	5.05	
2	20.53	5.01	4.85
4	20.58	5.07	4.83
5	20.65	5.16	4.72
6	20.85	5.17	4.75
8	21.	5.13	4.74
10	21.22	5.06	4.73
12	21.2	5.03	4.85
14	20.73	4.94	
16	20.35	5.03	4.83
18	20.3	5.13	4.70

Here the flattening is almost insensible at the mouth of the tube, and when $BS=$ about 14, and is greatest midway. As the first leap takes place when $BP=6.45$, and the second when $BP=20.78$, we may infer the wave of the reed to be 13.4, nearly, so that here the least effect is produced, when the pipe is nothing or the whole wave, and the greatest when it equals the half wave, as I had stated. The value of *BP* is very nearly constant, and its variations appear to follow a regular law, its mean value being nearly at the half waves.

Note E.

As this book is scarce, I have given the passage, which is curious, (*Facteur d'Orgues*, par D. Bedos de Celles 1766—70, p. 439.)

" On doit remarquer que lorsqu'on veut mettre un Tuyau d'Anche au ton qui lui est propre selon la longueur où il se trouve, on le fait monter en baissant la rasette (je suppose que celle-ci touchoit le coin,) le son devient mâle, harmonieux. Si l'on baisse un peu plus la rasette, le son devient plus doux, plus tendre, mais moins mâle et moins éclatant. Si l'on baisse encore la rasette, le son diminué, il s'éteint et devient sourd; si l'on baisse encore la rasette, le son double, c'est à dire, qu'il monte tout à coup d'un ton ou d'un tierce et quelquefois davantage; il change d'harmonie, et ce son ne vaut rien. On le fait redescendre en rehaussant la rasette, jusqu'à ce qu'il revienne à son vrai ton, qui doit-être mâle, èclatant et harmonieux, jusqu'à faire sentir un Bourdon qui parleroit ensemble avec le Tuyau d'Anche," p. 439.

.... " Pour les éprouver, on mettra la main dessus un instant tandis qu'ils parlent; comme si on vouloit les boucher, alors le Tuyau commencera à doubler; mais il se remettra de lui même au ton aussitôt qu'on aura oté la main. S'il ne se remet pas de lui même, ce sera une marque qu'il sera un peu trop long, &c."......p. 440.

Erratum.

Page 241, line 11, *for* with bass reeds, *read* with bass and tenor reeds.

XI. *On the Theory of the Small Vibratory Motions of Elastic Fluids.*

By J. CHALLIS, M.A.

FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *March* 30, 1829.]

1. Any one that has given much attention to the mathematical theory of sound, will be aware that notwithstanding the labours of the most eminent mathematicians, great obscurity is still attached to it. Much of this obscurity, I have been led to think, is owing to the manner in which *discontinuous functions* have been introduced into the subject; and as geometers of late have been more engaged in the use of them than in scrutinizing the evidence on which they rest, I will endeavour to state, as briefly as possible, the nature of this evidence. It depends, I believe, almost entirely on the authority of Lagrange, and on his two dissertations contained in the first and second volumes of the Miscellanea Taurinensia. His first ***Researches***, however admirable in other respects, cannot be adduced in reference to the point before us, because that part of them which bears upon it, contains a step in the proof which can by no means be admitted. In fact, it mainly depends on the sum of the series $\cos\theta + \cos 2\theta + \cos 3\theta +$ &c. *ad infinitum*, which he determines to be always equal to $-\frac{1}{2}$. And in truth, if the exponential expressions be put for the cosines, and the

series be summed to infinity, this result is obtained. But the objection is, that a mode of summing a converging series is applied to one which is not convergent. The only legitimate method is to sum the series to m terms, and to find what the sum becomes when m is infinite. Lagrange does this; he finds the sum to be $\frac{\cos m\theta - \cos(m+1)\theta}{2(1-\cos\theta)} - \frac{1}{2}$, and says, that the first term disappears when m is infinite, because the 1 may be neglected in comparison of m. But it cannot be admitted that two arcs, however great, which differ by a quantity θ, have the same cosines independently of the value of θ. The fallacy of the reason assigned for neglecting 1, will be apparent, by putting the sum of the series under this other form,

$$\frac{\cos\frac{m+1}{2}\theta\sin\frac{m\theta}{2}}{\cos\frac{\theta}{2}},$$

which does not give the same result as before, when 1 is neglected in comparison of m. I have adverted to this error, because in consequence of it, Lagrange exhibits to view a discontinuous function, the possibility of doing which, may well be called in question. It is not necessary to enquire how the reasoning may be conducted, if this step be corrected, because the second *Researches* are in principle the same as the first, and are not liable to a similar objection. In these he has elaborately, yet strictly shewn, as far as I have been able to follow the reasoning, that the motions he is in search of, are not subject to any law of continuity:—that the motions, for instance, at a given instant, in a column of fluid stretching between two given points, cannot be given generally by any known line or function. He supposes, therefore, that they will be given, by a *new* set of functions, neither algebraical, transcendental, nor mechanical,

but discontinuous *per se*, and by this property of discontinuity distinguished from every other. This definition has been admitted by all subsequent writers. But it deserves to be considered in what sense, and to what extent an investigation of this nature can demonstrate any property of functions. The science of quantity is a perfect science; it needs not the aid of any other, and exists prior to its applications to questions of nature, and independently of them. When in the applications, any form or property of functions is arrived at by the operations that are performed, it will always be possible to arrive at the same, by abstracting from the physical question, and performing the same operations by pure analytical reasoning. For in the applications, we are, in general, concerned about time, space, force, and matter,—ideas of a totally dissimilar kind, but possessing this in common, that we can conceive of them as consisting of parts, and in virtue of this common quality, after establishing a unit for each, we are able to express their observed relations numerically, or by lines or letters the representatives of numbers. All subsequent reasoning is then conducted according to the rules of analysis, and cannot possess a greater generality in regard to the modes of expressing quantity, than the operations conducted by those rules admit of. If an attempt be made to prove the existence of discontinuous functions by pure analysis, it will be impossible to succeed, because, as Lagrange says, "the principles of the Differential and Integral Calculus, depend on the consideration of variable algebraical functions, and it does not appear, that we can give more extent to the conclusions drawn from these principles, than the nature of these functions allows of. But no person doubts that in algebraic functions, all the different values are connected together by the law of continuity." (*Misc. Taur.* Tom. I. p. 21.) Accordingly, no discontinuous function can be *exhibited to view*. The inference to

be drawn with certainty from Lagrange's reasoning is, that if a number of particles, constituting a line of fluid, are in motion, the line which bounds the ordinates erected at every point, proportional to the velocities at a given instant, is not necessarily regular. It may consist of portions of continuous curves, connected together at their extremities, and be expressed analytically by a function, which possesses no distinctive property of discontinuity, but changes form abruptly and in a manner always given by the data of the problem to be solved. But if he had limited himself to this inference, and not supposed the existence of a new order of functions, he could not have determined the velocity of sound, and must have confessed that the analytical theory had not succeeded in solving that problem. For the demonstration he gives of it, rests altogether on the existence of discontinuous functions, such as they are above defined: and herein it differs entirely from Newton's solution of the same problem, which requires no new property of curves or functions, but deduces the velocity directly from the constitution of the medium:—a method, which certainly at first sight appears the more natural. As, however, we are sure that the velocity of the propagation of sound, must be a deduction from the principles on which the analytical investigation is founded, if no other mode of making the deduction can be thought of, we must be content to take up with discontinuous functions. No person can object to them who does not supply an equivalent, provided always they be considered in the present state of analytic science, not as demonstrated to exist, but as hypothetical, and like all hypotheses, established only by the extent and success of their applications. It was necessary to premise so much as this about discontinuous functions, in order to give a reason why any one, who treats of the vibrations of an elastic medium, has a right, if he can, to leave these functions out of

consideration; and that the best possible argument for their non-existence is, to shew how to do without them.

In the dissertation that follows, I have reasoned as if all functions were *per se* continuous, and setting out with this principle, have discussed the integrals containing arbitrary functions, prior to any supposition about the mode in which the fluid was put in motion; considering that as the investigation which led to these integrals was conducted without reference to any such supposition, and as they are consequently applicable to every point in motion, all inferences drawn from such discussion, must also apply to every point in motion. This method of treating the subject, dispenses with that of D'Alembert and Lagrange, who consider the differential equation of the motion, to be equivalent to an infinite number of equations of the same kind as itself, each of which applies to a single point. The first inference drawn from this manner of reasoning on the motions in space of one dimension is, that every point is moving in such a manner, as results either from a motion of propagation in a single direction, or two simultaneous motions of propagation in opposite directions. The velocity of the propagation is determined, and is, for air, the quantity commonly obtained by theory for the velocity of sound. Again, it is shewn that the forms of the functions are not entirely arbitrary, but limited by the nature of the question to a certain species, the primary form of which corresponds to the curve that occurs in Newton's reasoning, and by writers on the theory of vibrating chords called the *Taylorean Curve*. As any number of these curves will simultaneously satisfy the partial differential equations, it is inferred that the vibrations they indicate, may *co-exist*. If any portions of these curves, or of the curves resulting from the combination of any number of them, be joined together at their extremities, and so form an irregular line, every two consecutive ordinates

of which differ by an insensible quantity, as this line will satisfy the same differential equations, it indicates a possible motion, which is consequently of that bizarre and irregular kind, which Lagrange first demonstrated to be the general character of the vibrations. The particular form, however, of this line is given, when the particular mode of the disturbance which caused the motion is given. I have endeavoured to exhibit as clearly as possible, the mechanical reasons of this kind of motion.

In the next place, the bearing of the theory on the musical sounds produced in tubes, is briefly considered, and particular attention is paid to the mode in which the air vibrates in a tube open at both ends, because on this point, the view I have taken, leads to an inference which is at variance with the received theory.

The equation which gives the motion in space of two dimensions is integrated approximately, and the approximation is shewn to be such, that the integral will apply with accuracy to almost all cases that can occur. Euler's integral of the equation that applies to the motion in space of three dimensions, which has ever since his time been considered to be particular, is here shewn to be the proper general solution, and adequate to solve all the cases of small motions. This view of it is justified by its application to some problems of interest, particularly to oblique reflections, and the problem of resonances. In conclusion, I have stated as a result of the whole preceding investigation, the manner in which analysis points out the laws of any phenomena, the theoretical enquiry into which conducts to the solution of a partial differential equation.

I. *Motion in Space of one Dimension.*

2. To begin with the simplest case, let us suppose a portion of the medium to be inclosed in a very slender cylindrical tube

of indefinite extent, and to be unsolicited by any extraneous forces. Let v = the velocity of the particles at the distance x from a fixed origin, and at a time t reckoned from a given epoch; s = the condensation at the same distance, the mean density of the medium being = 1; and a^2 a constant proportional to its mean elastic force. The usual investigation leads to the equations,

$$\frac{d^2\phi}{dt^2} = a^2 \frac{d^2\phi}{dx^2} \dots\dots\dots\dots\dots (1),$$

$$\frac{d\phi}{dt} + a^2 s = 0 \dots\dots\dots\dots\dots (2),$$

$$v = \frac{d\phi}{dx} \dots\dots\dots\dots\dots\dots\dots\dots (3).$$

The integral of equation (1) is

$$\phi = F_{\prime}(x - at) + f_{\prime}(x + at).$$

Hence
$$v = F(x - at) + f(x + at) \dots\dots\dots (\alpha),$$

$$as = F(x - at) - f(x + at) \dots\dots\dots (\beta).$$

It is particularly to be observed with respect to these equations, that the origins of x and t are perfectly arbitrary; and that as the equations were investigated without reference to the manner in which the particles were put in motion, all results derived from them, prior to any hypothesis about the mode of disturbance, must be quite general; that is, must obtain in whatever way the particles have been caused to move, provided always that v be very small compared to a. Because each of the functions F and f, satisfies separately the differential equations, the motion which results from the consideration of either of them by itself, will be possible, though not the most general that can obtain. Suppose we consider the function F. Then $v = as = F(x - at)$. These equations shew that in the case sup-

posed, the velocity is always proportional to the condensation. Also if the curve be traced which gives the velocities and condensations at any instant, by supposing t constant and x variable, it is plain that the same curve is obtained, whatever be t; but it will be at different distances from the origin of x, according to different hypotheses made on the value of t. Also if t be supposed variable and x constant, the series of values which v and as take whilst t varies from t to $t+\tau$, are the same as are obtained, when t is supposed constant, by making x vary from x to $x - a\tau$. Hence, the velocities and condensations which the particles at a given point undergo during the time τ, are the same as those which the particles in a space $a\tau$, measured from the given point *towards* the origin of a, are undergoing at the instant τ commences. The motion will therefore be understood, by conceiving the curve which gives the velocities and condensations at every point, to move without undergoing alteration, along the axis and from the origin of x. The velocity of its motion will be found by making x and t vary at the same time, in such a manner that $F(x-at)$ does not alter. Hence,

$$d.(x-at)=0, \quad \frac{dx}{dt}=a.$$

This is the velocity of propagation; which is thus shewn to be constant, and in all cases the same. Considering now the function f by itself, we shall have $v=-as=f(x+at)$. A discussion similar to that applied to the function F, will prove that these equations indicate a series of velocities and condensations, which may be represented by a curve of invariable form, moving *towards* the origin of x with the uniform velocity a. For making $d\,(x+at)=0$, $\frac{dx}{dt}=-a$, which shews that the direction of propagation is opposite to what it was in the former case. Also v has changed sign,

as retaining the same sign. It follows from the preceding reasoning, that the motions of the particles are always such as result either from a motion of propagation in a single direction, or from two simultaneous motions of propagation in opposite directions.

3. In order to find out the nature of the functions F and f, conceive two propagations exactly alike, to produce the same motions of the particles in the same order, but to have contrary directions. There must be one point at least where the motions are constantly the same at the same instant, in virtue of the two propagations, but in opposite directions. Consequently v must $= 0$ at this point, whatever be t. If l be its distance from the origin of x, $F(l-at)+f(l+at)=0$. Describe two curves, the equation of one of which is $y=F(l-z)$, and of the other $y=f(l+z)$. Then because $F(l-z)=-f(l+z)$, for the same value of z the ordinates are equal with opposite signs. Hence, if one of the curves be transferred to the opposite side of the axis, the two will coincide. This transfer is made by changing the sign of y; consequently $-f(l+z)=F(l+z)$, and $F(l-z)=F(l+z)$. It appears thus, that the curves which F and f embrace must all satisfy the above condition, the meaning of which is, that a point may be taken in the axis of z, such that the ordinates at equal distances from it on each side are equal. Again, if two curves exactly alike, and possessing this property, move in opposite directions, there must of necessity be a position in which they coincide. Hence, f may be changed into F, and for all positions of the curves, $v=F(x-at)+F(x+at)$. Let the distance of a point at which $v=0$ whatever be t, from the origin, be $=l'$, a quantity related to l in a manner which will presently be determined, and not necessarily the same as l, because the origin of x is arbitrary. Hence, $F(l'-at)=-F(l'+at)$; a condition which all the curves given by F and f must satisfy, and which shews that

a point may be taken in the axis, such that the ordinates at equal distances on each side of it, are equal with opposite signs. At this point, the value of the ordinate is 0, because if z be made equal to zero in the equation, $F(l'-z) = -F(l'+z)$, $F(l') = -F(l')$, which cannot be, unless $F(l')=0$. Also at the point where $F(l-z) = F(l+z)$, the ordinate is a maximum or minimum; for $-F'(l-z) = F'(l+z)$, and making $z=0$, $F'(l) = -F'(l)$, which cannot be, unless $F'(l) = 0$. Hence l' and l are so related, that when they are measured from the same origin, the difference between them, is always equal to the interval between a maximum ordinate, and a point where the ordinate $=0$. The curve which satisfies all the above conditions, is described in Fig. 1. It must extend indefinitely in both directions, because it must be equally disposed on each side of the maximum ordinate OQ, and on each side of C, the point where the curve cuts the axis. AB, BC, CD, &c. are all equal, and the portions AaB, BbC, CcD, &c. are all exactly alike. Let $AB = \lambda$. This is the only parameter which the equation of the curve can contain. Hence

$$y = F(z) = \lambda \,.\, F\left(\frac{z}{\lambda}\right).$$

Also because the curve extends indefinitely in the positive and negative directions, $F\left(\frac{z}{\lambda}\right)$ must be some trigonometrical function of the abscissa, considered as an arc of a circle, the circumference of which is 2λ. The simplest that presents itself is

$$y = m\lambda \sin \frac{\pi z}{\lambda},$$

m being an arbitrary numerical quantity. The required conditions will also be satisfied in the most general manner by the equation

$$y = m\lambda \sin \frac{\pi z}{\lambda} + m'\lambda \sin \frac{3\pi z}{\lambda} + m''\lambda \sin \frac{5\pi z}{\lambda} + \&c.$$

because this equation, by reason of the unlimited number of terms, may be made to belong to any curve possessing the required properties, by properly disposing of the values of m, m', m'', &c. The primary form, however, in all these curves, is that given by the equation

$$y = m\lambda \sin \frac{\pi z}{\lambda},$$

and the motions they indicate, result from a composition of motions indicated by this curve, as will be proved in a subsequent Article. It is this relation between y and z, which occurs in Newton's famous forty-seventh Proposition. He proved that the excursion of a particle *may* follow the law of a vibrating pendulum, and in consequence that its velocity may vary as the sine of a circular arc, representing the time from the beginning of its motion, on the scale in which the whole circumference represents the time of an excursion in going and returning. It may here also be observed, that the reasoning by which we determined the velocity of propagation is fundamentally the same as Newton's, since the determination depends on the property $v = as$, on which Newton founds his reasoning in Props. 48 and 49. His views appear to be quite correct as far as they go: no one who objects to them can point out any thing that is absolutely erroneous, and every one who defends them, is willing to allow that they are incomplete.

It has been shewn, that when two motions exactly alike are propagated simultaneously in opposite directions, the state of the particles at any instant is determined by the equations,

$$v = F(x - at) - F(x + at),$$

$$\text{and} \quad as = F(x - at) + F(x + at),$$

and that there is a certain value l of x, for which

$$F(l - at) = F(l + at)$$

whatever be t, and a certain value l' for which

$$F(l' - at) = - F(l' + at),$$

whatever be t. The motion would be equally determined by the equations,

$$v = F(x - at) + F(x + at),$$
$$as = F(x - at) - F(x + at),$$

the origin of x being changed; but the former are the more convenient equations, because s does not change sign with the change of the direction of propagation, whereas v does. At the point, the abscissa of which is l, $v = 0$, and $as = 2F(l - at)$. This point is called a *node*. The other point, whose abscissa is l', is called a *loop*, and at it $as = 0$, $v = 2F(l' - at)$. It is plain that the loops as well as the nodes recur as often as the curve cuts the axis, and that each loop is separated from the adjoining node by the constant interval $\frac{\lambda}{2}$.

4. According to a remark before made, the preceding reasoning, being conducted without reference to the manner in which the fluid was put in motion, must be considered in a general point of view. The inference to be drawn from it is, that the motions of the particles in general, result from two motions of propagation obtaining simultaneously in opposite directions, that the velocity of propagation is always equal to the constant a, and that, considering the propagation in one direction only, the motions of the particles are in all cases *primarily*, (not necessarily,) such as are indicated by the curve $y = m\lambda \sin \frac{\pi z}{\lambda}$; in other words, their motions are always resolvable into a motion of this kind. The integral of a partial differential equation should always be subjected to a discussion like the foregoing, before any application is made of it, and for the purpose of directing us to the mode of making the application. Another remark, which it is important to make, is, that although the

curve $y = m\lambda \sin \frac{\pi z}{\lambda}$ extends indefinitely along the axis of z, it is not thence to be inferred, that the same particles go on oscillating backwards and forwards for an indefinite length of time. For the differential equations of the motion were deduced on the supposition that no forces acted on the fluid: consequently, the disturbances which put it in motion must be of the nature of impulses, which alter the relative positions of the particles, and by this change of position, call into play the fluid's elasticity, which is the immediate cause of the oscillatory motions, that are the subject of our consideration. When the disturbance ceases, the source from which the force is derived is stopped, and the motions of the particles must undergo a corresponding change, at first near the disturbance, then at points more remote, as the propagation proceeds. The form of the curve $y = m \sin \frac{\pi z}{\lambda}$, shews that the particles *may* oscillate alternately backwards and forwards, so as to return after every two oscillations to the same position; and this is a kind of motion, which is plainly compatible with the supposition of the existence of no force to cause a permanent motion of translation. But as it is by reason of the disturbance alone that the particles move at all, this periodicity in their motions, must answer to a like periodicity in the disturbance; and if the periods of the latter be limited to a certain number, the oscillations of a given particle will be limited to the same number. For whenever a particle has completed an oscillation, it is in a state in which it *can* remain permanently, if no extraneous force acts on the fluid, since its velocity is nothing, and the density where it is situated, is at the mean. If therefore the disturbance, at the end of one of its periods, suddenly stop, at the end of the corresponding oscillation of a particle, the elastic force which moves it, will suddenly

cease to be developed, and the particle will remain at rest. This is a particular case of that discontinuity of the motion, which was proved by Lagrange to obtain. In general, it may be concluded that any motion is possible, which is indicated by a portion of the curve $y = m \sin \frac{\pi z}{\lambda}$, included between any two points at which it cuts the axis; but in every case, the nature of the disturbance must determine between what two points the portion is to be taken; as will appear more plainly by examples hereafter to be adduced.

5. Let us now proceed to the application of the integral of the differential equation of the motion to particular cases. First, conceive a series of waves of the *primary* form to be generated at a given point, and to be propagated in the positive direction. Let us fix upon origins of t and x, and suppose the given point to be at a distance x' from the origin of x, and the commencement of the disturbance to be separated from the origin of t by an interval τ. Then the motion at any time t and distance x, will be given by the equations

$$v = as = m\lambda \sin \frac{\pi}{\lambda} . (x - x' - a . \overline{t - \tau});$$

for they satisfy the differential equation of the motion, and v and as each $= 0$, when $x = x'$, and $t = \tau$. It is not allowable to take x greater than $x' + at$, because at the end of t the propagation has not proceeded beyond this distance. If the propagation had been towards the origin, we should have had

$$v = - as = - m\lambda \sin \frac{\pi}{\lambda} . (x - x' + a . \overline{t - \tau}),$$

and x must not be taken less than $x' - at$.

As x' and τ are constant and arbitrary,

$$v = \pm as = \pm m\lambda \sin \frac{\pi}{\lambda} . (x \mp at + c),$$

according as the propagation is from or towards the origin of x. If in consequence of a limited duration of the disturbance, the number of oscillations of each particle be limited to n, the preceding equations are applicable at a given instant, only to particles included in a space $n\lambda$, all the others being at rest, and to a given particle only for an interval $\frac{n\lambda}{a}$, which is the duration of its motion. In general, having given the commencement and duration of a disturbance, supposed to cause a certain number of oscillations exactly, we can always determine what particles are affected by it at a given instant; also the beginning and end of the motion which a given particle undergoes.

Suppose now that there are several disturbances of the same kind as that above considered, and that some produce motion in the positive direction, some in the contrary. Having given the commencement and duration of each disturbance, it is required to find in what manner the particles are moving at any given instant. The differential equation $\frac{d^2\phi}{dt^2} = a^2 \frac{d^2\phi}{dx^2}$ will be satisfied if we put

$$\phi = F_1(x - at) + F_2(x - at) + \&c.$$
$$+ f_1(x + at) + f_2(x + at) + \&c.$$

each of the functions belonging to a separate disturbance. If we consider each function by itself, it will give us the motion which results from the disturbance to which it belongs. Hence the above equation shews that when a particle is affected by several disturbances simultaneously, the motion it receives is the resultant of all the different motions it would have, if each disturbance acted separately. And this is a general proof of the co-existence of small vibrations, in rectilinear propagated motion. The problem in question will therefore be solved, by putting

$$(\gamma)\quad V = m\lambda \sin\left(\frac{\pi}{\lambda}\,.\,\overline{x - at + c}\right) + m'\lambda' \sin\left(\frac{\pi}{\lambda'}\,.\,\overline{x - at + c'}\right) + \&c.$$

$$- m_1\lambda_1 \sin\left(\frac{\pi}{\lambda_1}\,.\,\overline{x + at + c_1}\right) - m_2\lambda_2 \sin\left(\frac{\pi}{\lambda_2}\,.\,\overline{x + at + c_2}\right) + \&c.$$

and in consequence,

$$(\delta)\quad aS = m\lambda \sin\left(\frac{\pi}{\lambda}\,.\,\overline{x - at + c}\right) + m'\lambda'\,.\,\sin\left(\frac{\pi}{\lambda'}\,.\,\overline{x - at + c'}\right) + \&c.$$

$$+ m_1\lambda_1 \sin\left(\frac{\pi}{\lambda_1}\,.\,\overline{x + at + c_1}\right) + m_2\lambda_2 \sin\left(\frac{\pi}{\lambda_2}\,.\,\overline{x + at + c_2}\right) + \&c.$$

Let s, s', &c. s_1, s_2, &c. be the condensations arising from the respective disturbances.

Then $\quad V = a\,(s + s' + \&c. - s_1 - s_2 - \&c.) = a\,(\sigma - \sigma'),$

$$aS = a\,(s + s' + \&c. + s_1 + s_2 + \&c.) = a\,(\sigma + \sigma').$$

Whenever $\sigma = \sigma'$, $V = 0$, and $S = 2\sigma$; and whenever $\sigma' = -\sigma$, $S = 0$, and $V = 2a\sigma$.

The equations (γ) and (δ) apply to the most general motion that can possibly take place, because the principle of the co-existence of small vibrations, coupled with the reasoning of Art. 3, shews that the vibrations of the particles must always primarily be such as the equation $y = m\lambda \sin \frac{\pi z}{\lambda}$ indicates, and therefore can in no case be other than those which arise from a composition of motions of this kind. As the terms of these equations are to be taken between certain limits, depending on the durations of the respective disturbances, which it is allowable to do, because when the terms are so taken, the expressions for v and as satisfy the equation $\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2}$, the line which defines the resulting velocities and condensations at a given instant, is not continuous, unless the disturbances be all supposed to continue

for an indefinite length of time. We shall first consider the case in which this line is continuous.

The propagation being supposed in the positive direction only, if also $c = c' = c'' = \&c.$, V will be 0 for any given value of t at a certain point, namely, that for which $x = at - c$. There will not be another point at which $V = 0$, unless $\lambda, \lambda', \lambda''$, &c. have a common multiple. Let $\lambda' = \frac{\lambda}{2}$, $\lambda'' = \frac{\lambda}{3}$, &c., and suppose also $c = 0$, as its value is arbitrary: hence,

$$V = aS = \lambda m \left\{ \sin \frac{\pi(x - at)}{\lambda} + \mu \sin \frac{2\pi(x - at)}{\lambda} + \mu' \sin \frac{3\pi(x - at)}{\lambda} + \&c. \right\}.$$

Here V and aS become 0 as often as $x = at + n\lambda$, and the curve which gives the velocities and condensations at a given instant, cuts the axis of x at points separated by the constant interval λ. Suppose $t = 0$; then the equation of the curve is

$$y = m\lambda \left\{ \sin \frac{\pi x}{\lambda} + \mu \sin \frac{2\pi x}{\lambda} + \mu' \sin \frac{3\pi x}{\lambda} + \&c. \right\}.$$

The general form of it is given in Fig. 2. The loop cd is exactly equal to ab, and symmetrically equal to bc. Also equal ordinates occur at points separated by the constant interval λ. The preceding is the general equation of the waves which produce musical sounds, when the disturbance arises out of the action of the parts of the fluid on one another, and follows the law of continuity: for the sole condition required in these waves is, that they recur at regular intervals. It is possible that this condition may be fulfilled at the same time that the loops ab, bc, cd, make up a discontinuous line; but in such cases, the disturbance which causes the waves must also be of a discontinuous nature.

6. Conceive two motions exactly alike to be propagated in

opposite directions along the cylindrical tube. There must be a point at which the velocities are the same and in the same order, in virtue of the two propagations, but in opposite directions. At this point therefore the particles will be at rest. The motion will in no respect be changed, if an indefinitely thin rigid partition be placed at right angles to the axis of the tube, just where the particles are stationary. The fluid will be divided into two separate columns, in each of which the motions will be the same as before. But plainly, the particles in one column cannot be affected by a propagation which has its source in a disturbance made in the other. Hence, the effect of such disturbance is supplied by *reflection* at the partition. It thus appears that the obstacle gives rise to a series of reflected waves exactly like the incident waves, and that the particles in contact with the reflecting body do not move.

7. I come now to a more particular consideration of the discontinuity of the motion, and for the sake of illustration shall begin with establishing a possible mode of generating the waves, which are all along supposed to be of the primary type. Conceive a series of these waves to be propagated along the tube, and at a certain point an indefinitely thin diaphragm the weight of which is insensible, to be placed, and to be capable of moving with perfect freedom in the direction of the axis. The motion of propagation will evidently proceed just as if the diaphragm did not exist, since it only acts as a means of transmitting the pressure of the particles on one side to those on the other. But as the particles on one side do not communicate with those on the other, the motion on the side looking in the direction of the propagation, will remain the same as before, if the diaphragm, by an independent cause, be made to oscillate just as it does by reason of the propagation. It will thus become a disturbing cause proper for generating a series of primary waves. By parity

of reasoning, a like series will be generated on the other side, and will be propagated in the contrary direction; and the two series will be so related, that the particles in contact with the diaphragm, will at every instant be just as much condensed or rarefied on one side, as they are rarefied or condensed on the other. Suppose the diaphragm to perform a single oscillation forwards. When it comes to rest, the velocity and condensation of the particles in contact with it will be nothing, and since it remains at rest, they will have no tendency to change their state. The same may be said of the state of the contiguous particles at the succeeding instant, and so on, through all the particles in the tube. Hence, the motion will be represented by a single loop of the curve $y = m\lambda \sin \frac{\pi z}{\lambda}$, situated on the positive side of the axis, and supposed to travel along the tube with the uniform velocity a. It follows that every particle will be shifted through a space, equal to that through which the diaphragm has been moved. If the tube be of a limited length, when the wave comes to its extremity, a small portion of fluid will issue from it, just equal to the quantity the diaphragm has displaced; and this plainly must be the case, before the equilibrium within the tube can be restored. No reason presents itself, for inferring that any portion will afterwards enter, or that any reflection will take place at the extremity. Our investigation as yet, cannot inform us what happens after the wave has forsaken the tube: we shall revert to this point in treating of propagation in space of three dimensions. On the other side of the diaphragm, a rarefied wave will be propagated just equal to the condensed one, and when the wave comes to the extremity of the tube, a portion of fluid will enter, just equal to that which issued from the other extremity;

8. Next consider what will happen, if the diaphragm, after

generating a portion *cd* (Fig. 3.) of a primary wave, suddenly stop. The particles in contact with it, which were moving with a velocity represented by *ac*, are suddenly reduced to rest, because they cannot separate for any appreciable time from the diaphragm, on account of the great force which presses them against it. Now in whatever manner the other particles are affected, we may be sure that as soon as the diaphragm ceases to move, the law of condensation given by the curve *cd* will begin to change, and the change will be accompanied by a movement of the particles regulated by the laws of propagation demonstrated in the foregoing Articles. By referring to Art. 6, it will be seen that the movement will proceed as if *reflection* were taking place at *ab*, and that the curve *cd* gives the compound condensation arising from the incident and reflected portions of the wave. Also the equations

$$v = F(x - at) + f(x + at),$$
$$as = F(x - at) - f(x + at)$$

shew, that where $v = 0$, that is, at the reflecting surface,

$$F(x - at) = -f(x + at), \quad as = 2\,F(x - at).$$

Hence, bisect *ac* in *e*, describe some curve *ef*, (the nature of which will hereafter be considered,) to represent the incident portion of the wave, and diminish the ordinates of *cd* by quantities equal to those of *ef*, so as to form the curve *en*. Then the particles will move in such a manner that the ordinates of the broken curve *end*, will represent the velocities in the direction *ad*, and the curve *ef* those in the contrary direction. The particles in contact with the diaphragm will have no motion at all, but will undergo the condensations indicated by double the ordinates of the curve *ef*. When the condensation becomes 0, these particles will have no more tendency to change their state,

and the wave will forsake the diaphragm. The motion will thenceforth be indicated by the uniform propagation of the curve $f'e'n'd'$, of which the portions $f'e'$ and $e'n'd'$, are exactly equal to fe and end. Or it may be indicated by the disjointed portions $f_{,}e_{,,}$ $m_{,}n_{,,}$ $c_{,}d_{,,}$ which possess the advantage of shewing the effect of suddenly stopping the diaphragm. As the motion represented by $f_{,}e_{,}$ is just equal and opposite to that represented by $m_{,}n_{,,}$ one destroys the other. Hence the motion of translation of each particle, is that which it would receive by going through the velocities represented by the curve $c_{,}d_{,,}$ and is therefore just equal to that of the diaphragm; as it plainly ought to be.

We may gather from what precedes, that a particle may go through velocities represented by a portion of the primary curve, such as $n_{,}d_{,,}$ only one extremity of which is situated in the axis: and it matters not what is the subsequent motion, because the movement is equally possible if a loop pqr, terminated at one end in r, the foot of the ordinate to n', be superadded to the line $f'e'n'd'$, and the motion be such as will be given by a line traced by taking ordinates equal to the sum of the corresponding ordinates of these two lines. As the dimensions of the loop are arbitrary, the resulting line is also arbitrary. We may now shew that a particle may perform the motion indicated by any portion whatever of the primary curve, the prior and subsequent motions being arbitrary. And first it is obvious, that as the particles are susceptible of the motion which results from the propagation of the curve $fend$, (Fig. 4.) in the direction fd, they are equally susceptible of that which results from a propagation of the same curve in the contrary direction. For at any given instant, the condensations are the same and in the same order in the two cases, and, consequently, the particles have the same relative positions. Therefore if these positions be possible in one case, they must be so in the other.

But as the order of condensations at a given point is reversed when the direction in which the curve travels is changed, the succession of the velocities which a given particle undergoes, must also be reversed. At the same time the direction of its motion is changed. Now that this may take place, will appear by considering that if $fp = x$, $pq = y$, the accelerative force acting on a particle situated where the condensation is y, varies as $\frac{dy}{dx}$, and is therefore the same, and directed in the same way, in the two propagations. Hence, f being the same function of s in the two motions, we shall have

$$\frac{vdv}{ds} = f, \text{ and } v^2 = 2\int f ds + C,$$

an equation embracing both motions, and shewing that for the same value of s, v has two equal values, one positive, the other negative, one belonging to the motion forwards, the other to the motion backwards. This being premised, let a diaphragm be made to move so as to generate the wave *fend*, and when the portion *fenc* is generated, let it be suddenly stopped. The wave will in consequence assume the type $f'c'n'nef$. And it may be shewn precisely as before, that a particle may perform the motion indicated by the portion $n'n$ of the primary wave, whatever be its prior and subsequent motions.

Suppose now any number of curves to be described, the general equation of which is,

$$y = m\lambda \sin \frac{\pi(x+c)}{\lambda} + m'\lambda' \sin \frac{\pi(x+c')}{\lambda'} + m''\lambda'' \sin \frac{\pi(x+c'')}{\lambda''} + \&c.$$

and which differ according to different hypotheses made on m, λ, c, &c. Then it follows from what precedes, and from the principle of the co-existence of small vibrations, that if any

number of portions of one or more of these curves be taken such, that being translated to a common axis, and maintaining relatively to it the positions they had with respect to their own axes, they are capable of uniting at their extremities so as to form an irregular line, the ends of which are situated in the axis, this line will designate in the most general manner, the state of particles subject to motion propagated in a single direction. It will be seen that the equation $V = aS$ always obtains, and that the particles pass consecutively through the same states. Two such lines moving with the uniform velocity a in opposite directions, will represent the most general motion of which the fluid is susceptible.

9. It has been shewn in the preceding Article, that the effect of suddenly stopping the diaphragm, was to change the velocities proportional to the ordinates of cn, into velocities proportional to the difference of the ordinates of en and ef, (Fig. 5.). Hence, describing a curve cf, the ordinates of which are double the corresponding ordinates of ef, the effect was to diminish the velocities of the particles included in a certain extent af, by quantities proportional to the ordinates of cf. Therefore if the fluid be at rest, and at its mean density, and the diaphragm be suddenly made to move with the velocity ac, it will shake the particles to an extent af, and cause them to commence moving with velocities, the law of which will be given without sensible error by the curve cf. We have now to find the nature of this curve. Suppose ac, which represents the velocity given by the impulse to the particles on which it immediately acts, to be some function of af, the distance to which its effect extends;

$$ac = Y, \quad af = X, \quad Y = f(X).$$

But as the particles at p are affected just as if a diaphragm were placed there, and suddenly made to move with

the velocity pq, if $fp=x$, $pq=y$, $y=f(x)$. Hence the line fc is given by a single equation, which is always the same whatever be the velocity impressed, and the angle at which it cuts the axis, is independent of this velocity. Also since the particular form of cf must result from the constitution of the fluid, its equation will be

$$y = m\lambda \sin\frac{\pi x}{\lambda},$$

or more generally,

$$y = m\lambda \sin\frac{\pi x}{\lambda} \pm m'\lambda' \sin\frac{\pi x}{\lambda'} \pm m''\lambda'' \sin\frac{\pi x}{\lambda''} + \&c.$$

But as there is nothing from the conditions of the question to determine λ, λ', λ'', &c. because they are independent of the impressed velocity, and as these quantities cannot be considered arbitrary, since the equation is of a determinate nature, they must be assumed so as to be made to disappear. This will obviously be effected by making each of them indefinitely great. Hence,

$$y = \pi\,(m \pm m' \pm m'' + \&c.)\,x = kx,$$

and cf is a straight line. It follows that the extent af over which the shock is felt, varies as the velocity communicated to the particles immediately acted upon. The momentum communicated will vary as af × the mean of the velocities of the particles, that is, as ac^2. Hence if a diaphragm at rest, be suddenly moved with the velocity V, the resistance to be overcome in the first instant, varies as V^2. Let $af=l$, and $l=2nV$; this quantity n may be determined experimentally. For suppose m^2 equal to the area of the surface of the diaphragm, μ equal to its mass, and let it be put in motion by the impact of a mass M, moving with the velocity w, so that the two masses M and μ with the fluid in contact with the diaphragm, begin to move immediately after the impact with the velocity V. Let δ be the density of the

fluid; then the mean velocity it receives is $\frac{V}{2}$, and the momentum communicated to it

$$= m^2 l\delta \cdot \frac{V}{2} = m^2\delta n V^2.$$

Hence, $$Mw = (\mu + M) V + m^2\delta n V^2,$$

$$n = \frac{Mw - (M+\mu) V}{m^2\delta V^2};$$

$$\text{and } l = \frac{2}{m^2\delta}\left(M\frac{w}{V} - \overline{M+\mu}\right).$$

There would probably be great practical difficulty in determining V, but the difficulty might be diminished by choosing m^2 very large, which may be done without affecting the accuracy of the determination.

10. Because v is a function of x and t,

$$\left(\frac{dv}{dt}\right) = \frac{dv}{dt} + \frac{dv}{dx} \cdot \frac{dx}{dt}.$$

Let the propagation be in a single direction: then

$$v = F(x - at); \ \frac{dv}{dt} = -a \cdot F'(x - at) = -a\frac{dv}{dx}.$$

$$\text{Hence,} \quad \left(\frac{dv}{dt}\right) = \frac{dv}{dt}\left(1 - \frac{v}{a}\right), \ \text{for } \frac{dx}{dt} = v.$$

But $\frac{v}{a}$ may be neglected in comparison of 1, as has been done before. Therefore the accelerative force acting on a particle at the distance x from the origin, very nearly

$$= \frac{dv}{dt} = \frac{d^2\phi}{dtdx}.$$

But ϕ is by hypothesis very small, and the reasoning is not accurately applicable to cases in which it exceeds a certain

limit. Therefore also the values of $\frac{d^2\phi}{dtdx}$ must range below a certain small limit. And since

$$\frac{d^2\phi}{dtdx} = -a^2 \cdot \frac{ds}{dx},$$

if the condensations be given by

$$y = m\lambda \sin \frac{\pi x}{\lambda},$$

the accelerative force corresponding to the condensation y, varies as

$$\frac{dy}{dx}, \text{ or } m\pi \cos \frac{\pi x}{\lambda},$$

and is greatest when

$$x = 0, \text{ and } \frac{dy}{dx} = m\pi.$$

Hence m can never exceed a certain small quantity. Moreover as the condensation $as = -\frac{d\phi}{adt}$, and that the greatest value of y is $m\lambda$, it follows that $m\lambda$ must not exceed a certain limit. But the value of λ is not limited. These considerations exclude those forms of the primary curve in which the maximum ordinate is not very small compared with λ. In general when the motion is given by an irregular line, the angle which two consecutive tangents to it make with each other, must not exceed the greatest value $m\pi$ admits of, and the ordinates to the line must not surpass the limiting value of $m\lambda$. If λ be supposed very great, and

$$x = \frac{\lambda}{2} \pm z, \quad y = m\lambda \sin\left(\frac{\pi}{2} \pm \frac{\pi z}{\lambda}\right) = m\lambda \cos \frac{\pi z}{\lambda} = m\lambda$$

very nearly, shewing that a portion of the curve becomes a straight line parallel to the axis. Hence the motion indicated by this line is possible. Also when λ is very great, $y = \pi m x$, and

by taking m positive and negative from 0 up to its limiting value, we get a number of straight lines, all of which belong to possible motions. Hence the motion may be such as is shewn by two straight lines ac, cb, (Fig. 6.) inclined differently to the axis and meeting at c. The whole of the preceding reasoning is applicable *mutatis mutandis* to vibrating chords, and this line may consequently represent that form of the chord, which, being practically possible, led Euler to start the idea of discontinuous functions to account for it theoretically. D'Alembert, unwilling to admit into analysis any thing so singular, thought that the theory was inapplicable to cases of this kind. By taking an unlimited number of small portions of the straight lines included in the equation $y = \pi m x$, it would be possible to form a segment of any continuous curve by joining them together;—for instance, a segment of a circle;—and the motion indicated by such a line would be possible, provided always the ordinates be small, and at every point the tangent be inclined at a small angle to the axis.

It thus appears that the form of the functions F and f, is not *necessarily* that which I have called *primary:* they may take any other forms whatever, subject to the limitations abovementioned. These forms are given when the particular mode of the disturbance is given; or, since the disturbance may always be supposed to be made by the motion of a diaphragm, the form of the function is given when the particular motion of the diaphragm is given. It is very necessary to know what is the primary form of these functions, in all cases in which the motion to be considered, results from the action of the parts of the fluid on one another. All that has been said from Art. 7. will serve to shew that it is by reason of the discontinuity of the motion, that the functions are susceptible of other forms, which correspond to an action of a different kind, as for

instance, to the action of a solid on the fluid. It must be remembered, that the proper proof of the discontinuity of the motion consists in the circumstance, that the motion is to be determined by the integration of a *partial* differential equation, the peculiar property of which is, that it may be satisfied by giving a series of values, linked together by no law of continuity, to that variable which is considered a function of the other two. The demonstration of this property must be conducted by pure analysis. It has been given by Lagrange; for, as I believe, the reasoning on this point, in the second volume of the *Misc. Taur.* may be abstracted from the physical question with which it is involved.

11. Lastly, we have to consider what happens when the diaphragm goes on moving uniformly with the velocity with which it was made to commence. Take ac (Fig. 7.) to represent the impressed velocity; af, the distance over which its influence is felt at the first instant; and join cf. The ordinates of this line, as has been shewn, will represent the velocities initially impressed on the particles included in af. Bisect ac in e, take ae' in ca produced equal to ae, and join ef, $e'f$. The motion in the first instant will be the same as if two waves, designated by ef, $e'f$, the former condensed, the latter rarefied, were moving simultaneously in opposite directions. ef will be moving in the direction af, $e'f$ in the contrary direction, and the velocities of the particles at the same distance from a, will be the same and in the same direction in both, so that the line cf will represent the compound velocities. As the motion proceeds, the wave $e'f$ will be reflected, and ef will go on in its original direction. Also the diaphragm will generate condensations proportional to its velocity. Hence the motion after a given time may be represented in the following manner. Let τ be the given time. Take aa' (Fig. 8.) the space over which the

propagation travels in the time τ, and $a'c' = ac$, to represent the velocity of the diaphragm; so that the area $a'c'ca$ will represent the condensations generated in the time τ. Take $fq = 2 \times aa'$, draw the ordinate pq, and pt parallel to $a'q$. Let qs make the same angle with df as ef does, produce fe to b, and join pc'. Since $qa' + c'c = 2 \times aa' + aq = af$, and that $ec = ea$, it follows that $c'b = a's$. But the triangle pbt is in every respect equal to $qa's$; therefore also $bt = a's$, and $c't$ is bisected in b. Hence $c'p$ is parallel to cf, and wv is bisected in e. As the condensed and rarefied waves travel with equal velocities in opposite directions, their extremities in the time τ will have separated by qf, and $qa's$ will be the portion of the rarefied wave not reflected, $c'ceb$ the reflected portion. Hence the condensations $cc'a'a$ are diminished by $cc'be$, and also by $arsa'$, or by its equal $evtb$, and the remaining condensations are $avta'$. Again, the condensations $peaq$ are diminished by qar, or by its equal pve, which leaves remaining $pqva$. Therefore at the given time the condensations are given by the thick line fpt. Also since the direction of the velocities in the rarefied wave is changed by reflection, the velocities $a'c'ca$ will be diminished by $c'bec$; but they will be increased by $arsa'$, or by its equal $ewc'b$; and the velocities on the whole will be $awc'a'$. Again, the velocities $qpea$ are increased by qar, or by its equal pew, becoming on the whole $qpwa$. Hence at the given time the velocities are given by the thick line fpc'. Because $c'p$ is always parallel to cf, it follows that the particles at any distance from the diaphragm move uniformly with the velocity first impressed on them, till they acquire the condensation corresponding to their velocity. When the motion has continued long enough for the rarefied wave to be totally reflected, that is, during a time $\frac{af}{a}$, it will become such as Fig. 9. represents, $c'ef$ being the line which gives both the condensations

and velocities. Afterwards it will be represented for any length of time by a straight line $c'z$, parallel to the axis and distant from it by $d'c'$; for this motion was shewn in the preceding Article to be possible. Hence it appears, that if a piston be made to move uniformly in an open cylinder, the resistance it meets with from the air varies as its velocity, if the velocity be small compared with that of sound. Also as the rarefaction on one side of the diaphragm is just equal to the condensation on the other, it may be concluded that if a column of air be made to move uniformly through a cylindrical tube, it will either be condensed or rarefied proportionally to its velocity. As this is a case in which no extraneous force acts, the equation $v = \pm as$ should apply.

If the diaphragm after a while suddenly stop, the resulting wave will take the type, $fecc'e'f'$, (Fig. 10.). If it move but for a very short time, $fq'qf'$ (Fig. 11.) will be its type. In these two cases $mm' = a\tau$, τ being the time during which the diaphragm moves, and the area $mm'n'n$ represents the condensations it generates. Reasoning exactly analogous to all that is contained in this Art., will apply to the case in which the diaphragm produces rarefaction.

By what precedes we are informed what will take place, if a partition, which separates a column of condensed or rarefied fluid from fluid of the mean density, be suddenly removed; for the action of the fluid will be the same as if the partition were suddenly converted into a moveable diaphragm.

As we have now considered the effects of impact, and of pressure on the fluid, and as every disturbance must be made by the one or the other of these means, or by both together, the foregoing discussion will suffice for determining the effect produced by any disturbance, the exact nature of which is given.

II. *Application to the Vibrations in Musical Tubes.*

12. I proceed to say a few words on the application of the theory to sounds produced in musical tubes. Conceive a series of waves of the primary type to be generated at the open end A of a tube closed, at the other end B, and to be propagated from A towards B. When they arrive at B they will be reflected back, will return to A, and will be there affected just as they would be, if the end B were removed, the tube were prolonged to A', so that $BA' = BA$, and the waves proceeded to an open extremity A'. Hence a tube closed at one end, and a tube of double the length of the other, open at both ends, comport themselves alike in respect to the propagation of waves. After reflection two waves, indicated by the two curves cb, $c'b'$, (Fig. 12.) exactly equal, will meet; and in consequence at some point m the velocity will be always equal to 0. Let t be reckoned from the time at which c, c', were simultaneously at m. Then $cm = at$; and if

$$mp = x, \quad pq = y, \quad y = m\lambda \sin\frac{\pi}{\lambda}.(x + at).$$

$$\text{Also if } pr = y', \quad y' = m\lambda \sin\frac{\pi}{\lambda}.(at - x).$$

$$\text{Hence, } v = y - y' = m\lambda \left\{\sin\frac{\pi}{\lambda}(x + at) + \sin\frac{\pi}{\lambda}(x - at)\right\}$$

$$= 2m\lambda \sin\frac{\pi x}{\lambda}\cos\frac{\pi a t}{\lambda},$$

$$as = y + y' = m\lambda \sin\frac{\pi}{\lambda}(x + at) - \sin\frac{\pi}{\lambda}(x - at)$$

$$= 2m\lambda \cos\frac{\pi x}{\lambda}\sin\frac{\pi a t}{\lambda}.$$

As the particles at the closed end must necessarily be at rest, we may reckon x from B towards A, and date t from an instant at which the condensation at B is 0. Then v will be equal to 0, wherever $x = n\lambda$, and as will $= 0$, wherever $x = (n + \frac{1}{2})\lambda$. The points obtained by putting $n = 0, 1, 2, 3$, &c. are in the first case nodes, in the other loops. As these nodes and loops arise *solely* from the reflection from the closed end, they cannot exist in the tube open at both ends, at least not from the same cause, and the two tubes in this respect are not circumstanced alike. We have seen reasons in Art. 7, and shall give others hereafter, to conclude, that the waves on returning back to the open extremity, will be transmitted into the circumambient fluid, causing alternations of rarefaction and condensation in it, as frequent as those generated in the tube. And as in this manner there will be no cause at the open extremity of the closed tube to affect its nodes and loops, so there will be none at the extremity of the tube open at both ends to produce nodes and loops. Our conclusion is also confirmed by the fact that when the cause which produces the aerial vibrations ceases, the sound to which they give rise ceases to all appearance instantaneously:—an effect which could scarcely take place if the particles were reduced to rest solely by their inertia, and by their friction against the sides of the tube. The vibrations of the air in a cylindrical tube are analogous to those of an elastic chord, and the closed ends correspond to the fixed points of the chord. If no point be fixed in the direction in which the original motion impressed on the chord is travelling, it will go on interruptedly without being reflected; so that according to the view I have taken, as there is no difference in theory between the two cases, there will be none in practice.

13. It does not appear from any thing that precedes, that we can *a priori* assign a set of waves which a tube of given

length will transmit rather than all others. Since, therefore, it is found that tubes produce a certain series of notes in preference to all others, the cause is to be sought in the mode and circumstances of the disturbance, and unless these be exactly known, the fact cannot be explained theoretically. It is necessary to distinguish between those cases in which the vibrations arise out the elastic nature of the fluid itself, as for instance, when they are caused by blowing across a hole; and the cases in which the vibrations are immediately impressed on the fluid by an elastic substance, as in Reed Organ-Pipes. To the former I shall direct my attention, having in view the experiments described by Biot. (*Traitè de Physique*, Tom. II.) When a musical sound is caused by a continuous and equable disturbance of this nature, we may conclude, because it is musical, (see Art. 5.) that the type of the wave is given generally by the equation,

$$y = \lambda \left\{ m \sin \frac{\pi x}{\lambda} + m' \sin \frac{2\pi x}{\lambda} + m'' \sin \frac{3\pi x}{\lambda} + \text{\&c.} \right\}.$$

Suppose the disturbance to be made at the extremity of a tube open at both ends. Experience shews that when the lowest note possible, *the fundamental note*, is sounded, λ is equal to the length of the tube. This note, which may be called 1, and is expounded by the first term, is heard, because the coefficient m of this term, becomes by the nature of the disturbance, large compared to those of the other terms; not however to the exclusion of the sounds 2, 3, 4, &c. which correspond to these terms, for they are heard as *harmonics* with the first. By a change of circumstances each of the coefficients m', m'', &c. may be made in order prominent above the rest, and the sounds 2, 3, 4, &c. be generated. Accordingly in practice it is found that this effect is produced by altering the disturbance *in degree*, its particular nature remaining the same. (Biot, pp. 125, 138.) M. Biot states, (p. 131.) that he has ascertained in the most con-

vincing manner, that when any note n of the series 1, 2, 3, &c. is sounded, all the $n-1$ lower notes are heard with it. This also our theory would lead us to expect, inasmuch as it ascribes the sounding of any note to the prominence of the coefficient of the term corresponding to it above those of the other terms. Admitting therefore, as a matter of experience which we have not sufficient data to account for theoretically, that λ is always equal to the length of the tube, theory is sufficiently accordant with fact.

The above theory does not require us to know the precise mode of disturbance, in order to account for the series of notes. I will, however, venture to suggest that when a musical sound is caused by blowing over the orifice of a tube, the disturbance is created by the contiguity of a stratum of air in motion, and therefore, as would appear from Art. 11., condensed or rarefied, to the stratum of air kept at rest by the sides of the orifice, and therefore of mean density. As the condensation is proportional to the velocity, a large and slow current would produce a grave note, a small and rapid one, a high note.

With respect to the tube closed at one end, we may at once infer from what has been said in Art. 12., that its fundamental note, and all its other notes are the same as those of a tube open at both ends of double the length, if the disturbances be made in the two cases under the same circumstances. But till we know every thing that is concerned in producing the vibrations in the two tubes, we cannot pronounce with certainty when this identity of circumstance exists. It would seem that the density of the fluid at the open end of the stop-tube must be the mean, and consequently that this must be the position of a loop. The series of notes will therefore be 1, 3, 5, &c.

There is reason to think that the vibration of the tube itself, is an element entering into the determination of λ, for it has

been observed, that if a *tin* tube be applied to the ear, its fundamental note will at all times be distinctly heard, without any apparent cause, but if a paper tube be applied, no note is heard. But whatever be the causes which define the length of λ, they appear not to be of a very strict nature, for it is possible by skill and practice, to modify the notes of the series 1, 2, 3, 4, &c., and to produce notes intermediate to them. (Biot. pp. 128, 130.) When in the common theory, the values of λ are accounted for by saying that the density of the air at the extremities of the tube must be the mean, to be in equilibrium with the external air, a cause is assigned, which is perhaps of too determinate a nature. Besides that it seems in contradiction to the fact, that the sound ceases immediately that the cause which produces it ceases. M. Poisson felt the force of these objections, and a considerable portion of his Memoir (*Acad. Scien.* Ann. 1817.) is employed in obviating them. The above theory, by ascribing the length of λ entirely to the mode of disturbance, leaves a latitude in this respect, and admits the possibility of inserting notes between those of the regular series. When the tube has holes in the side, like a flute, λ is seldom equal to the length, but appears to be determined both by the side holes, and by that at the extremity. The fundamental note being so determined, an alteration of the disturbance in degree, will produce as in other cases, the series 1, 2, 3, &c. in order. (See Lambert, *Berlin Memoirs*, 1775.) And here I will observe, that supposing the condensations and rarefactions generated at one end of the flute, to chase each other towards the other end, the effect of a side hole will be to diminish them by a quantity depending on its size, and in such a manner that the type of the waves after passing the hole is different from their previous type, only in having the ordinates diminished in a given ratio. (Art. 11.) The condensations and rarefactions act on the external air at the

side holes just as they do at the extremity, and thus it is that the waves which are separately generated in the external air at the side holes and at the extremity, produce sounds of the same pitch and quality. It would be difficult to explain this fact according to the received theory.

Upon the whole, our theory seems to prove that the *series* of notes is attributable to the nature of the fluid, but that the particular value of λ is due to causes, independent of the fluid, which are not yet satisfactorily understood.

III. *Motion in Space of two and of three Dimensions.*

14. Next let the motion take place in space of two dimensions. The equation for this case is,

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right).$$

$$\text{And } a^2 s + \frac{d\phi}{dt} = 0, \quad p = \frac{d\phi}{dx}, \quad q = \frac{d\phi}{dy},$$

p and q being the velocities in x and y respectively. To integrate the above equation, suppose $x^2 + y^2 = r^2$. Hence

$$\frac{d\phi}{dx} = \frac{d\phi}{dr} \cdot \frac{x}{r},$$

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2} \cdot \frac{x^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right),$$

$$\frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2} \cdot \frac{y^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{y^2}{r^3}\right),$$

$$\therefore \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2} + \frac{d\phi}{rdr},$$

$$\text{and } \frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dr^2} + \frac{d\phi}{rdr}\right) \ldots\ldots\ldots\ldots (A).$$

$$\text{Now } \frac{d^2.\phi\sqrt{r}}{dr^2} = \sqrt{r}\left(\frac{d^2\phi}{dr^2} + \frac{d\phi}{rdr}\right) - \frac{\phi}{4r^{\frac{3}{2}}},$$

$$\therefore \sqrt{r}.\frac{d^2\phi}{dt^2} \text{ or } \frac{d^2.\phi\sqrt{r}}{dt^2} = \frac{d^2.\phi\sqrt{r}}{dr^2} + \frac{\phi}{4r^{\frac{3}{2}}}.$$

This equation is integrable whenever $\frac{\phi}{4r^{\frac{3}{2}}}$ may be neglected, that is, when r is not small, because ϕ is a very small quantity. The integral is,

$$\phi\sqrt{r} = F_{,}(r - at) + f_{,}(r + at);$$

$$\therefore\ as = -\frac{d\phi}{adt} = \frac{1}{\sqrt{r}}\{F(r-at) - f(r+at)\},$$

$$\text{and } v = \sqrt{p^2 + q^2} = \frac{d\phi}{dr}.$$

$$\text{Hence } v = \frac{1}{\sqrt{r}}\{F(r-at) + f(r+at)\} - \frac{\phi}{2r^{\frac{3}{2}}}.$$

$$= \frac{1}{\sqrt{r}}\{F(r-at) + f(r+at)\},$$

as $\frac{\phi}{r^{\frac{3}{2}}}$ has been before neglected. The interpretation of this integral will be understood by what will be said of the integral of

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dx^2}\right),$$

which applies to motion in space of three dimensions. To obtain the latter integral, put $x^2 + y^2 + z^2 = r^2$. A process like the preceding will lead to the exact equation

$$r\phi = F_{,}(r - at) + f_{,}(r + at) \ldots\ldots\ldots\ldots (B).$$

$$\text{Also } a^2s + \frac{d\phi}{dt} = 0,\ p = \frac{d\phi}{dx},\ q = \frac{d\phi}{dy},\ u = \frac{d\phi}{dz};$$

$$\therefore\ v = \sqrt{p^2 + q^2 + u^2} = \frac{d\phi}{dr}.$$

$$\text{Hence}\ \ as = \frac{1}{r}\{F(r - at) - f(r + at)\},$$

$$v = \frac{1}{r}\{F(r - at) + f(r + at)\} - \frac{\phi}{r^2},$$

and $\frac{\phi}{r^2}$ may be neglected whenever r is not small. The integral (B) was long ago obtained by Euler, and has more recently been arrived at in a different manner by M. Poisson, (*Mem. Acad. Scien.* 1818.) who considers it a particular case of the general integral. But nothing presents itself in the above solution to forbid our concluding that it gives the *proper* general integral of the differential equation. It may be no objection to our conclusion, that this able analyst has expressed the general integral under a less simple form, by the method of Definite Integrals, because that method, as he says, is not unique and determinate, and may not therefore be very proper for ascertaining the simplest form of the general integral. The use that will be made of the integral will elucidate this point. The equations $x^2 + y^2 = r^2$, and $x^2 + y^2 + z^2 = r^2$, shew that in both cases r is the distance of the point under consideration from the origin of co-ordinates. Equations (A) and (B), shew that in both ϕ is a function involving r and t alone. Therefore since

$$as = -\frac{d\phi}{adt}, \text{ and } v = \frac{d\phi}{dr},$$

v and as are functions of r and t alone. Hence may be inferred that *primarily* the motion of every particle is at a given instant some function of its distance from a point in a line drawn through it in the direction of its motion. The generality of this inference is legitimate, both because nothing was said about the manner

in which the fluid was disturbed, in the investigation of the equations of the motion, and because the origins of r and t are quite arbitrary. Also because v and as will have the same values for a given value of r and t, in whatever direction r be drawn from the origin, supposing the forms of the arbitrary functions to be the same for all directions, it follows that the general character of the motion is spherical; that each particle may be considered as moving in the direction of the radius of a sphere, and its motion to be some function of that radius.

15. As in obtaining the equations

$$v = \frac{1}{\sqrt{r}}\{F(r-at)+f(r+at)\},$$

$$as = \frac{1}{\sqrt{r}}\{F(r-at)-f(r+at)\},$$

for the motion in space of two dimensions, and the equations

$$v = \frac{1}{r}\{F(r-at)+f(r+at)\},$$

$$as = \frac{1}{r}\{F(r-at)-f(r+at)\},$$

for the motion in space of three dimensions, only terms of the order $\frac{\phi}{r^2}$ were neglected, it follows that these equations are applicable to most cases that can occur; for the above general character of the motion shews that $\frac{\phi}{r^2}$ will almost always be a very small quantity. We shall confine our attention at present to the latter equations. By reasoning upon them exactly as we reasoned on the analogous equations in rectilinear propagation, the following deductions will be made:—

1st. The velocity of propagation is always equal to a.

2nd. The function F applies to propagation from a centre, the function f to propagation towards a centre.

3rd. When the propagation is entirely from the centre,

$$v = as = \frac{1}{r} F(r - at),$$

when entirely towards the centre

$$v = - as = \frac{1}{r} f(r + at),$$

4th. The primary form of the functions F and f is

$$m\lambda \sin \frac{\pi(r - at)}{\lambda}.$$

The same reasoning as before about the discontinuity of the motions is applicable.

The possibility of the motion towards a centre is proved by experiments on air, which shew that surfaces of a certain shape may by reflection concentrate sound in a focus. In like manner, if a slight agitation be made at the centre of the surface of water in a circular basin, the wave emanating from it, after being reflected at the side of the basin, will return to the centre again.

15. Suppose a series of waves to be propagated in such a manner, that the velocities and condensations shall be equal at all equal distances from a fixed point. This may be conceived to be effected by means of an elastic globe placed in the fluid, and made to expand and contract in a determinate manner, and in the same degree in all directions from its centre. The equations applicable to the motion are,

$$v = as = \frac{1}{r} F(r - at + c).$$

And if the radius of the globe and law of its expansion and contraction be given, the exact form of the function F will be known. For simplicity, suppose the globe to expand through a small space, then contract to its original dimensions, and remain at rest, so as to generate a single wave. The form of F, ascertained in the first instant, by the given motion of the globe, will be the same for all the particles subsequently affected by the disturbance; for there are no data whereby a change of form can be determined. And if we conceive the globe to be a hollow shell, filled with the fluid, by its contraction and expansion a motion of propagation *towards* the centre will be impressed on the fluid in its interior; the equations of the motion will be,

$$v = -as = -\frac{1}{r} F(r + at + c');$$

and the same reasoning as before applies to the constancy of the form of F. This is a general proof that there is no cause resident in the constitution of the fluid, to alter the direction of propagation; for spherical propagation either from or towards a centre, has been shewn to be the general law of the motion. In consequence of a disturbance of the kind above supposed, the particles in motion at any given instant will be included in a spherical shell, the thickness of which may be called the *breadth* of the wave. This breadth remains the same throughout the motion because the form of F remains the same. As the propagation is uniform, r and t may vary so that $r - at$ shall be equal to a constant a. Hence

$$v = as = \frac{F(a)}{r},$$

shewing that the velocities and condensations at corresponding points of the same wave, at different distances from the centre,

vary inversely as the distances. Let r be equal to the internal radius of a spherical shell of fluid, forming a part of the wave, $\delta =$ its thickness, supposed very small, and $1+s$ its density: then its mass $= 4\pi r^2 \delta (1+s) = 4\pi r^2 \delta$ very nearly; and its velocity varies as $\frac{1}{r}$. Hence its *vis viva* is the same at whatever distance it be from the centre, if δ be the same. Hence also the *vis viva* of the whole wave is constant during its propagation, because its breadth is constant. The same thing is easily proved of waves propagated in space of two dimensions.

16. Although the waves propagated from a disturbance made at a single point are always bounded by a spherical surface, because the velocity of propagation is always equal to a, the velocities and condensations of the particles are not the same in all directions, unless the disturbance be similarly related to all the parts of the surrounding space, as in the instance adduced in the preceding Art. The motion will be given in general by

$$v = as = \frac{1}{r} F(r - at),$$

and the form of F, when applied to the same wave, will be always the same in the same direction, but will be different in different directions, according to a law depending entirely on the nature of the disturbance. Conceive a pyramid the vertical angle of which, formed by four equal plane surfaces, is indefinitely small, to be placed with its vertex at the point of disturbance. The particles within the pyramid will perform their motions as they would if it were removed, because they move in lines directed to its vertex, and the sides of the pyramid, supposed indefinitely thin, supply the pressure which upon its removal would be exerted by the contiguous fluid. Now the motion within the pyramid is strictly such, that the velocities and condensations at all equal distances from the vertex are equal, and vary inversely as the dis-

tances. Hence supposing the number of the pyramids indefinitely great, and the velocities and condensations at equal distances to be given at a given instant by any law, depending on the initial disturbance, it will be possible to calculate the circumstances of the motion at any instant. Again, though the general character of propagation in space of three dimensions is spherical, it is not necessary that the boundary of a wave should be a spherical surface. It may be any surface whatever: but we infer from the general property, that each very small portion of the wave will obey the laws of spherical propagation:—it will move uniformly in the direction of its normal, as if it were a portion of a spherical wave having for its centre the centre of curvature. It is easy to see that the centres of curvature will be fixed points in space, and that the surface, whatever it may be at first, will continually approach to that of a sphere. The reasoning in this Art. is exactly analogous to that which is stated at the end of Art. 10.

The principle of the conservation of *vis viva* will hold in every wave, whatever be the shape of its boundary or the law of its type, because it holds for every individual portion of it.

17. Let us now make an application of the general solution to an instance in which the disturbance is of a very general nature. Suppose given disturbances to act at any number of points in space for any length of time: it is required to find what will be the consequent motion at a given instant of a particle in a given position. We will assume fixed rectangular axes and an origin of x, y, z; and date t from a fixed epoch. Let

$$x_1, y_1, z_1; \quad x_2, y_2, z_2; \quad x_3, y_3, z_3, \text{ \&c.}$$

be the co-ordinates of the points of disturbance; τ_1, τ_2, τ_3, &c. the intervals between the commencements of the disturbances and the beginning of t. Then the solution of the question will be effected by the equation,

$$\phi = \frac{F_1 \{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} - a(t+\tau_1)\}}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}$$

$$+ \frac{F_2 \{\sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2} - a(t+\tau_2)\}}{\sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2}}$$

+ &c. to as many terms as we please.

For each of these terms satisfies separately the differential equation of the motion, and gives the motion resulting from the disturbance to which it belongs. Also the terms taken collectively satisfy the same equation, and give the motions consequent upon all the disturbances acting simultaneously. We have thus a proof of the principle of the co-existence of small vibrations, and infer from it, that the motion which a particle has at any instant in consequence of several disturbances, is the resultant of the several motions it would receive, if each disturbance acted alone exactly as it does contemporaneously with the others.

18. Conceive two points to be disturbed under circumstances precisely similar, so as to be the origins of two series of waves exactly alike. Then, by the Proposition just proved, the particles in a plane bisecting at right angles the line joining these two points, will move entirely in this plane, and will have no motion perpendicular to it. If, therefore, in the place of the imaginary plane be substituted an indefinitely thin rigid partition, the motions of the particles will not be altered. But plainly the disturbance on one side of the partition cannot affect the particles on the other. Hence the effect of such disturbance is supplied by reflection at the partition. The angle of incidence is equal to the angle of reflection, and the reflected waves are exactly what the incident would have been, if they had gone on without interruption. We are thus informed, in what manner a series of waves acts dynamically on any obstacle whatever that

it meets with. The particles in contact with the obstacle glide along its surface, and do not affect it by their motion, but act solely by alternate condensations and rarefactions.

19. From reflection at a plane that at curved surfaces is easily deduced. I will only observe, that convex surfaces increase the curvature of a wave, concave surfaces diminish it; that if the incident waves be musical, the reflected waves will be musical also and of the same pitch; and that the type of each portion of a wave at any time after reflection, differs from the type of the corresponding portion before reflection, only in having all its ordinates altered in the same proportion. This law of the alteration of the type is as universal as the law of spherical propagation, of which it is a consequence. Hence if a series of waves be generated at any point of the fluid, and their type be given at first by $y = F(x)$, by whatever devious path they come to the ear, the type will on reaching it be $y = mF(x)$. It follows also from the manner in which reflection takes place, as shewn in the preceding Art., that the changes of density at a given point of the drum of the ear, on which the waves are incident, whether obliquely or not, are proportional to $F(at)$. Thus if the type be originally

$$y = m\lambda \sin \frac{\pi x}{\lambda},$$

when the waves reach the ear it will be

$$y = \mu\lambda \sin \frac{\pi x}{\lambda},$$

and the condensations at a given point of the drum will vary as $\mu\lambda \sin \frac{\pi a t}{\lambda}$. This will explain how it is that the ear distinguishes with accuracy, not only the pitch, but less marked shades of distinction, in sounds which are due to waves that

must have travelled from their origin by very irregular paths. The *intensity* of the sound is proportional to μ.

The velocity of propagation along any path whatever, is equal to a, because neither reflection nor the action of the parts of the fluid on one another alters this velocity.

From the preceding considerations it will not be difficult to perceive, that the motion along the line of propagation from one point to another, may be the same as if the fluid were contained in a tube, the transverse section of which is every where indefinitely small, but different at different points, and the axis of which coincides with the line of propagation. The position of the axis, and the law, either continuous or not, according to which the transverse section varies, will be known from the data of the problem to be solved, coupled with the laws of spherical propagation demonstrated in Arts. 15 and 16. We shall have

$$v = as = mF(\sigma - at) \dots\dots\dots\dots (\epsilon),$$

where σ may be measured along the axis from a fixed point in it, instead of being taken of arbitrary length, along different straight lines drawn in the direction of propagation, to all of which the axis is a tangent. The form of F is given by the initial disturbance, and m is inversely proportional to the radius of the tube, supposing the transverse sections to be circles. If the sections be every where the same, m is constant, and $v = as = F(\sigma - at)$, as in straight cylindrical tubes.

The equations (ϵ) apply to the motion which takes place along the axis of any straight musical pipe of finite length, even when prolonged beyond its mouth into the exterior fluid. For this line must be a line of propagation, as no reason exists why there should be deviation from it in one direction rather than another. It would be difficult to ascertain the alteration

m undergoes just as the wave leaves the mouth, but considerations like those of Art. 11. might enable us to do it. The constancy of *vis viva* must at least approximately obtain, and this alone is sufficient to shew that a series of waves equal to the incident cannot be reflected at an *open* mouth in the way D. Bernoulli supposed.

If a trumpet-shaped tube, the form of which is a surface of revolution be fitted to the cylindrical tube, the boundary of that portion of the wave which emanates from its surface, will be a surface generated by the revolution round the axis of the tube, of an involute of the curve which generates the surface of the trumpet; for the surface of the wave, where it meets the surface of the trumpet, must be perpendicular to it, since the direction of the motion of the particles, which is always at right angles to the former, must be along the latter. The condensations will be greatest in the direction of the axis, as well from the manner in which the wave spreads laterally, as because in this direction the curvature of the wave is least. Hence a trumpet sounds loudest in the direction of its axis.

It has been remarked by Euler, that no stop-pipe is musical unless it be cylindrical; and the fact may be explained by what was said in Art. 13, where it appeared that the density of the air at the open extremity must always be the mean. This cannot be if the tube be of any other shape than that of a cylinder, because the line of propagation from the mouth to the mouth again, will be less along the axis than in any other direction, if the closing end be a plane perpendicular to the axis. But if it be made to fit the boundary of the wave, theory shews that the pipe may be musical.

20. Because $v = as$ for motion in space of three dimensions, the resistance which a projectile whose velocity is small in comparison of a, encounters from the *pressure* of the air varies as

its velocity. But as it passes through the medium, it strikes at every instant against fresh particles, and the resistance from this cause is as the square of the velocity. (See Art. 9.) Hence upon the whole, the resistance to a projectile is partly as the velocity and partly as the square of the velocity. This is probably a near approximation to the law which obtains in nature. In small vibratory motions like those of chords, we may safely say that the resistance is as the velocity.

Suppose a series of primary waves to impinge perpendicularly on a vibrating chord, the diameter of which is small compared to 2λ the breadth of a wave. According to the law of reflection demonstrated in Art. 18. the air in contact with every point of that half of the chord which looks towards the source of the waves, will suffer condensations in virtue of reflection *alone*, proportional to $\sin \frac{\pi a t}{\lambda}$; the air in contact with the other half being supposed not affected. But it is plain that by reason of the gliding of the particles along the surface of the chord, the condensations will be established all round it. The law they will follow may be thus found. Whether the wave impinge on the chord, or the fluid be at rest and the chord oscillate just as a particle of the wave does, the effect as to the distribution of the condensations will be the same. In this latter case, each point of the surface of the chord will at every instant generate a condensation proportional to its velocity in the direction of the normal. Hence if $abcd$ (Fig. 13.) be the transverse section of the chord; OA the direction of its motion; $bf = Af = \delta$, the condensation at b at a given instant, on the scale in which $Of = D$ the mean density; the density Op corresponding to any angle θ reckoned from OA, will be $D + \delta \cos \theta$. In the case of the chord stationary, Od will be the mean density, and Op, the radius-vector of $deAe'd$, will become $D + 2\delta \cos^2 \frac{\theta}{2}$; for had it not been

for the obstacle presented to the wave by the chord, the density at the given instant would have been Oe. The quantity δ used above, varies as $\sin \frac{\pi a t}{\lambda}$, and it is easily shewn that the resultant of the pressures on the chord varies in the same manner. This being premised, it will be possible to solve the problem of sonances in the particular case in which the waves are incident perpendicularly on the resounding body. It will be supposed that every particle of it tends to return to its position of rest by the same force, varying as the distance from this position; and the supposition is allowable if it be permitted to infer from what has been said of the types of waves, that the primary form of vibrating chords is the ***Taylorean Curve.*** Every point of the chord will thus be acted upon by three forces; its own elasticity, the pressure of the impinging wave varying as $\sin \frac{\pi a t}{\lambda}$, and the fluids resistance varying as the velocity. Hence,

$$\frac{d^2 s}{dt^2} + p\frac{ds}{dt} + ns - m \sin \frac{\pi a t}{\lambda} = 0$$

is the equation to determine the motion; p and m being small in comparison of n. This equation being integrated, and the constants determined so that $s = 0$, and $\frac{ds}{dt} = 0$, when $t = 0$, the equation for $\frac{ds}{dt}$ will come out after all reductions,

$$\frac{ds}{dt} = \frac{\frac{\pi a}{\lambda} m}{\left(n - \frac{\pi^2 a^2}{\lambda^2}\right)^2 + \frac{p^2 \pi^2 a^2}{\lambda^2}} \cdot \left\{ \begin{array}{l} \left(n - \frac{\pi^2 a^2}{\lambda^2}\right) \cos \frac{\pi a t}{\lambda} + \frac{\pi a p}{\lambda} \sin \frac{\pi a t}{\lambda} \\ -e^{-\frac{pt}{2}} \left\{ \left(n - \frac{\pi^2 a^2}{\lambda^2}\right) \cos ht + \left(n + \frac{\pi^2 a^2}{\lambda^2}\right) \frac{p}{2h} \sin ht \right\} \end{array} \right\};$$

where h is put for $\sqrt{n - \frac{p^2}{4}}$. The value of m is constantly the same for the same point of the chord, but different at different points.

As this equation is exact, it follows that in all cases the body acted upon will execute simultaneously two sets of vibrations, one depending on its elasticity, the other on the disturbing cause. For the equation may be put under the form

$$\frac{ds}{dt} = Q\left\{\sin\left(\frac{\pi a t}{\lambda} + \theta\right) - e^{-\frac{pt}{2}} \cdot q \sin(ht + \theta')\right\},$$

But the latter term will quickly decrease on account of the factor $e^{-\frac{pt}{2}}$, and the oscillations will soon become isochronous, and will be of the same duration as those of the particles of the incident waves. This may in some degree explain why the drum of the ear is susceptible of the vibrations corresponding to any musical note, especially as in this instance n is probably not very large compared to p and m. On account of the small value of m, these oscillations must be excessively small, and not adequate to produce the phenomenon of resonances. But $\frac{ds}{dt}$ will be much larger when n is equal to, or nearly equal to $\frac{\pi^2 a^2}{\lambda^2}$; than in any other case, because the coefficient Q becomes great on account of the smallness of p. This is just the condition which experience shews to be fulfilled when resonance takes place; for $\frac{2\lambda}{a}$ = the time of oscillation of an aerial particle $= \frac{2\pi}{\sqrt{n}}$ the time of oscillation of the chord. In strictness the time of oscillation $= \frac{2\pi}{\sqrt{n - \frac{p^2}{4}}}$, (Whewell's ***Dynam.*** p. 206.) because the equation determining the motion of the chord is

$$\frac{d^2 s}{dt^2} + p\frac{ds}{dt} + ns = 0,$$

when no extraneous force acts on it. Suppose therefore $\frac{\pi a}{\lambda} = h$,

and for simplicity put $\frac{p}{4 \cdot \frac{\pi a}{\lambda}} = \tan \delta$. It will be found then that,

$$\frac{ds}{dt} = \frac{m}{p} \left\{ \left(1 - e^{-\frac{pt}{2}}\right) \cos \delta \sin \left(\frac{\pi a t}{\lambda} + \delta\right) - 2e^{-\frac{pt}{2}} \sin^2 \delta \sin \frac{\pi a t}{\lambda} \right\}.$$

The first term in the brackets, commencing from 0 will quickly increase, and the other, always small by reason of the factor $\sin^2 \delta$, will rapidly decrease, and the vibrations will soon become isochronous. Also they may be of considerable magnitude, because the ratio $\frac{m}{p}$ is not in general small. After a short time,

$$\frac{ds}{dt} = \frac{m}{p} \cos \delta \sin \left(\frac{\pi a t}{\lambda} + \delta\right).$$

The quantity n depends on the length of the resounding chord, or a submultiple of its length. The cause producing the waves may be a vibrating chord, the length of which is equal to that of the other, or a submultiple of its length, or lastly a multiple of it, and in this last case the resonances are caused by the *harmonic waves* of the vibrating chord.

On similar principles would have to be calculated the effect of a series of waves, propagated along the interior of a musical pipe, in producing those vibrations of it, which determine the *timbre* or quality of the note, and which I suspect are principally concerned in fixing the value of λ. (See Art. 13.)

21. In conclusion I will state the inference to be drawn from all that has gone before respecting the application of the integrals of partial differential equations to physics.

As the integral of a partial differential equation contains arbitrary functions, if we consider it in a purely analytical sense, it may be satisfied by any one of the infinite number of functions we can form at pleasure, or by any number of them combined, or lastly by the combination of portions of any number of them,

connected together by no law of continuity. If therefore the quantity sought after in any physical question, be given by the solution of a partial differential equation, this circumstance is itself a sufficient proof that the quantity is not subject to the law of continuity: it is a proof too that several quantities of the kind sought for may coexist. But of the infinite number of functions that will satisfy the differential equation, there will in general be a certain species which belongs in a peculiar manner to the question, and is to be determined by a discussion similar to that with which we commenced our consideration of the vibrations of an elastic fluid. No general rule can be given for such a discussion; the nature of the question itself must decide the manner of conducting it. It is essential that this primary form of the arbitrary functions be ascertained, before any application be made of the integral.

The views contained in this paper, I have found to be greatly confirmed by similar reasoning applied to the general equations of the motion of incompressible fluids.

J. CHALLIS.

Trinity College,
March 30, 1829.

Fig. 1

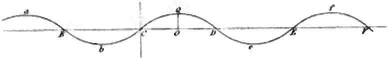


Fig. 2

Fig. 3

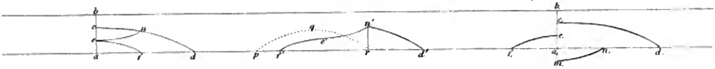


Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

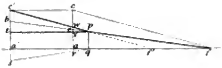


Fig. 9

Fig. 10

Fig. 11

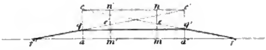


Fig. 12

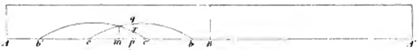
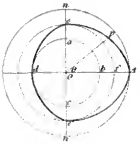


Fig. 13



XII. *On the Comparison of various Tables of Annuities.*

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[Read *March* 30, 1829.]

1. In last May I transmitted to the Philosophical Society of Cambridge some remarks upon the construction of Tables of Annuities; my object in that paper was to shew how the probabilities upon which Annuities depend, should be deduced from Tables of mortality, and I gave in illustration some Tables of Annuities calculated from observations of the mortality at Chester, by Dr. Haygarth, which appear to have been made with very great care. I have since compared these Tables with a great many others, and I now present the result of this comparison.

2. Very few registers of mortality give the deaths at every year throughout life, they generally give the deaths between birth and 5, 5 and 10, 10 and 20, 20 and 30 and so on for every decade. When the deaths are given between birth and 5, the *living* at 5, at 20, 30, &c. are known, and in order to form a complete Table of mortality it is necessary to interpolate the number of living at each intermediate age.

If the probability of an individual aged m years, living n years be called $p_{m,n}$, if r is the rate of interest, and if the same hypothesis of probability be adopted as in my former Paper which amounts to increasing by 1 the deaths at every age,

$$p_{m,n} = \frac{\text{living at } n + 101 - n}{\text{living at } m + 101 - m},$$

the value of a payment of unity after n years is

$$\frac{p_{m,n}}{(1+r)^n},$$

and the value of an annuity is

$$\Sigma \left\{\frac{p_{m,n}}{(1+r)^n}\right\},$$

m being constant in this expression and n variable.

Instead however of interpolating values of $p_{m,n}$ between those values which are known, it is better to interpolate at once between the values of $p_{m,n} \times (1+r)^{-n}$ which are given, but even this labour is unnecessary, because $\Sigma \frac{p_{m,n}}{(1+r)^n}$, or the value of the annuity is a function of those terms only of the series which are given.

Let $y_0, y_i, y_{2i}, \ldots y_{ni}, y_{(n+1)i}$, &c.
be successive values of any variable y,

$$y_0 = y_0,$$

$$y_1 = y_0 + i\,\Delta y_0 + \frac{i \,.\, i-1}{1\,.\,2}\,\Delta^2 y_0 + \text{\&c.}$$

$$y_{2i} = y_0 + 2i\,\Delta y_0 + \frac{2i \,.\, 2i-1}{1\,.\,2}\,\Delta^2 y_0 + \text{\&c.}$$

$$y_{ni} = y_{ni},$$

$$y_{(n+1)i} = y_{ni} + i\,\Delta y_{ni} + \frac{i \,.\, i-1}{1\,.\,2}\,\Delta^2 y_{ni} + \text{\&c.}$$

$$y_0 + y_i + y_{2i} + y_{3i} \ldots + y_m + y_{(n+1)i} + \text{\&c.} + y_{(mni-1)}$$
$$= n\,(y_0 + y_{ni} + y_{2ni} \ldots\ldots\ldots + y)$$
$$+\, i\,\{1 + 2 + 3 \ldots\ldots\ldots n-1.\}\,\{\Delta y_0 + \Delta y_{ni} + \text{\&c.} + \Delta y_{(m-1)ni}\}$$
$$+\, \frac{i}{1\,.\,2}\,\{1\,.\,i-1 + 2\,.\,i-2 + \text{\&c.} + n-1\,.\,1\}\,\{\Delta^2 y_0 + \Delta^2 y_{ni} + \text{\&c.}\},$$

$$\Delta y_0 + \Delta y_{ni} + \&c. + \Delta y_{(m-1)ni} = y_{mni} - y_0,$$

$$\Delta^2 y_0 + \Delta^2 y_m \ldots\ldots + \Delta^2 y_{(m-1)ni} = \Delta y_{mni} - \Delta y_0,$$

$$\Delta^3 y_0 + \Delta^3 y_m \ldots\ldots + \Delta^3 y_{(m-1)ni} = \Delta^2 y_{mni} - \Delta^2 y_0,$$

$$\text{when } ni = 1,\ i = \frac{1}{n}$$

the sum of the series is equal to

$$n\,(y_0 + y_1 + y_2,\ \&c.\ldots\ldots\ldots + y_{m-1})$$

$$+ \frac{1}{n}\{1 + 2 + 3 \ldots\ldots\ldots\ldots + n-1\}\{y_m - y_0\}$$

$$- \frac{1}{1.2n^2}\{1.n-1+2.n-2+3.n-3\ldots.+n-1.1\}\{\Delta y_m - \Delta y_0\}$$

$$+ \frac{1}{1.2.3n^3}\{1.n-1.2n-1+2.n-2.2n-2\ldots.$$

$$+ n-1.1.n+1\}\{\Delta^2 y_m - \Delta y_0\} + \&c.$$

The coefficient of $\Delta^q y_m - \Delta^q y_0$ is equal to the coefficient of x^{q-1} in the development of

$$\frac{(1+x)^{ni}-(1+x)}{1-(1+x)^i},$$

or, in other words, if this coefficient be called z_q,

$$\frac{x\{(1+x)^{ni}-(1+x)\}}{1-(1+x)^i}$$

is the generating function of z_q, and since $ni = 1$,

$$\frac{(1+x)^{ni}-(1+x)}{1-(1+x)^i} = \frac{x}{(1+x)^i - 1}$$

$$= \frac{1}{i} - \frac{i-1}{2i}x + \frac{i-1.i+1}{12i}x^2 - \frac{i-1.i+1}{24i}x^3 + \&c.$$

$$= n + \frac{n-1}{2n}x - \frac{n-1.n+1}{12n}x^2 + \frac{n-1.n+1}{24n}x^3 + \&c.$$

The sum of the series is

$$n(y_0 + y_1 + y_2, \&c. + y_{m-1})$$

$$+ \frac{n-1}{2}\{y_m - y_0\} - \frac{n-1 \,.\, n+1}{12 \,.\, n}\{\Delta y_m - \Delta y_0\}$$

$$+ \frac{n-1 \,.\, n+1}{24 \,.\, n}\{\Delta^2 y_m - \Delta^2 y_0\} + \&c.$$

Laplace has given in the 4th Volume of the *Mécanique Céleste*, p. 206, the particular value of this series which obtains when the interval i which separates the values of y is indefinitely diminished.

In this case the coefficient of $\Delta^q y_m - \Delta^q y_0$ is found by integrating

$$\frac{i \,.\, (i-1)(i-2) \ldots\ldots\ldots (i-q)\, di}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots q+1},$$

from $i=0$ to $i=n$, if $n=1$, the sum of the values of y or the area of the curve between y_0 and y_m

$$= \tfrac{1}{2}y_0 + y_1 + y_2 \ldots\ldots\ldots + \tfrac{1}{2}y_m$$

$$- \frac{1}{12}\{\Delta y_m - \Delta y_0\} + \frac{1}{24}\{\Delta^2 y_m - \Delta^2 y_0\}.$$

In applications of the former series to the calculation of annuities, reversionary payments, &c. y_m, Δy_m, $\Delta^2 y_m$, &c. $= 0$.

The first term in the series of the values of y or y_0 is the value of a present payment $= 1$, if we neglect the term

$$\frac{n-1 \,.\, n+1}{12 \,.\, n}\{\Delta y_m - \Delta y_0\},$$

and the following, and suppose the values of the annual payments to be in arithmetical progression, the value of an annuity on the life of a person aged 20, to commence at the end of the first year.

If $n = 10,\ y_0 = 1,\ y_1 = \frac{p_{20,\,10}}{(1+r)^{10}},$

$$= 10\left\{1 + \frac{p_{20,\,10}}{(1+r)^{10}} + \frac{p_{20,\,20}}{(1+r)^{20}} + \&c.\right\} - \frac{9}{2} - 1,$$

$$= 10\left\{\frac{p_{20,\,10}}{(1+r)^{10}} + \frac{p_{20,\,20}}{(1+r)^{20}} + \&c.\right\} + \frac{9}{2},$$

the values of annuities at 0, 5, &c. may be obtained in a similar manner. This value of the annuity will be a very close approximation, the error whatever it be, will be nearly constant for different tables of mortality, and as the first correction which is in this case $\frac{9}{2}$ is constant, the whole correction may be considered as constant. It may therefore be determined easily by calculating the annuity first accurately, and afterwards by the approximate method from any Table of mortality in which the deaths are given for every age, the difference between the two values so obtained will be the correction required. By means of the Chester Table for males, I determined the correction as follows, supposing the Table of mortality to give the living at 0, 5, 10, 20, 30, 40, 50, &c., and that the annuity commences at the end of the first year.

Age.		Age.	
At Birth	2.481	30	4.109
5	3.692	40	4.024
10	4.167	50	3.920
20	4.242	60	3.792

Thus the value of an annuity at 20 is

$$10 \times \left\{\frac{p_{20,\,10}}{(1+r)^{10}} + \frac{p_{20,\,20}}{(1+r)^{20}} \ldots\ldots \&c.\right\} + 4.242.$$

How close an approximation this method gives may be seen in Table II, where I have placed underneath the results which

I have obtained, those which have been obtained by other writers. The same series, p. 324, line 2, shews that the value of an annuity of £1. paid half-yearly, is the value of the same annuity paid yearly $+\frac{1}{4}$, and the value of an annuity of £1. paid weekly, is the value of the same annuity paid yearly $+\frac{51}{104}$, the annuity being supposed to commence at the end of a year, and the first weekly payment to commence at the end of a week.

When the Table of mortality which is made use of gives the deaths at every age, the preceding method can only be considered as an approximation, but in all cases I believe the error due to this method will be less than the error due to the errors of the observations.

The same series, p. 324, line 2, furnishes a method, which I think is the simplest which can be proposed, of calculating approximately the values of annuities or insurances on two or three lives.

The value of an insurance on one life, is easily deduced from the value of the annuity; in fact, if A is the value of the annuity, the value of the insurance in a single payment is

$$\{1 + A\}\frac{1}{1+r} - A,$$

and the value of the premium is

$$\frac{1}{A(1+r)} + \frac{1}{1+r} - 1.$$

When the persons observed upon whom the Table of mortality is founded, are few in number, and the deaths are given for every year, they will present considerable irregularities owing partly to the effect of accidental causes, and partly to the unavoidable errors of the observations, but these causes may be considered in theory as identical. If e, be the probability that

Age.	arlisle.		Chester.	Mean.	Age.
	Bills 1759—17	From the Bills.			
0	1000	1840	4066	......	0
5	426	1028	2434	......	5
10	373	939	2268	......	10
20	325	861	2105	......	20
30	272	765	1852	......	30
40	212	676	1608	......	40
50	147	558	1310	......	50
60	96	455	1019	......	60
70	52	282	651	......	70
80	17	130	243	......	80
90	2	32	41	......	90
100		4		......	100
0	1000	1000	1000	1000	0
5	426	558	598	631	5
10	373	510	557	585	10
20	325	467	517	542	20
30	272	415	455	472	30
40	212	367	395	404	40
50	147	303	322	329	50
60	96	247	250	247	60
70	52	153	160	150	70
80	17	70	59	55	80
90	2	7	10	5	90
100		2			100
	20.07	30.612	32.154	32.637	

Age.	London Bills of Mortality. 1757—1768.	1818—1827.	Paris.	Mean.	Difference between Mean and Chester.	Value of Difference in Days Purchased.	Age.
0	10.018	15.230	15.158	14.878	.021	7	0
5	18.763	21.318	21.720	21.729	.303	83	5
10	19.121	21.060	20.223	21.609	.373	101	10
20	17.208	18.459	18.061	19.519	.427	116	20
30	15.091	15.815	17.427	17.681	.482	131	30
40	12.960	13.408	15.216	15.309	.502	137	40
50	11.240	10.846	12.385	12.525	.844	231	50
60	8.856	7.548	9.139	9.267	.886	242	60
70	6.435	6.182	6.687	6.703	.401	109	70

Age.	Br	Paris.	Age.
0	7.	24644	0
5	4.	16869	5
10	4.	15880	10
20	3.	14633	20
30	3.	12129	30
40	2.	10064	40
50	2.	8172	50
60	1.	6174	60
70	.	3714	70
80	.	1093	80
90	.	92	90
100	.	1	100
0	)	1000	0
5	)	682	5
10	.	644	10
20	;	593	20
30	;	492	30
40	)	408	40
50	;	331	50
60	;	250	60
70	;	150	70
80	,	44	80
90	,	3	90
100			100
	40	34.145	

Age.	Brussels.	Amster	Mean.	Difference between Mean and Chester.	Value of Difference in Day's Purchase.	Paris.	Age.
0	13.851	14.2	15.282	.470	128	15.811	0
5	21.410	21.5	22.496	.147	40	21.320	5
10	21.126	21.0	22.385	.026	7	20.787	10
20	18.533	18.2	20.293	.071	19	18.482	20
30	17.046	16.6	18.371	.285	78	17.198	30
40	14.633	14.3	16.151	.416	113	15.224	40
50	12.011	11.5	13.368	.220	60	12.456	50
60	8.991	8.8	9.993	.679	186	8.977	60
70	6.473	6.8	6.723	.539	147	5.456	70

TABLE VI.

	A	B	C	D
Ratio of Increase of the Births.	1.005	1.010	1.015	1.005

TABLE O the Century previous to
the C was 1.005, and that the
Death

Age.	ES.		FEMALES.	
	Li	Decre^{s}.	Living.	Decre^{s}.
0	10	98	1720	129
1	8	81	1591	124
2	7	76	1467	120
3	7	76	1347	120
4	6	70	1227	121
5	6	71	1106	121
6	6	71	985	122
7	6	65	863	123
8	6	59	740	123
9	6	54	617	107
10	61	47	510	74
11	61	42	436	52
12	61	35	384	41
13	6	36	343	30
14	6	30	313	30
15	6	29	283	31
16	5	30	252	31
17	5	30	221	31
18	5	24	190	32
19	5	24	158	32
20	5	17	126	26
21	5	7	100	26
22	5	7	74	14
23	5	7	60	8
24	5	7	52	8

Calc—

Tables of Mortality.

Ratio of increase of the Births.		1	)0	Apparent. 1.005	1.010	1.015	1.020
Births.	10000	2675	)0	10000	10000	10000	10000
Between	Living.		2	5986	5638	5142	4636
0 and 5	40384	1080!2		5577	5158	4621	4089
5 — 10	33047	884(3		5177	4664	4104	3553
10 — 20	62745	1678!0		4554	3983	3425	2883
20 — 30	57613	1541:0		3954	3359	2834	2326
30 — 40	51296	1372:6		3221	2647	1986	1751
40 — 50	44237	1183·7		2506	2011	1619	1264
50 — 60	36080	965·4		1601	1238	984	753
60 — 70	26391	706(3		597	448	349	267
70 — 80	15139	404(3		100	97	93	57
80 — 90	5338	142'6					
90 — 100	1544	41.					
	373814	10000					
			77	33.039	28.713	25.143	22.084

Calculated upon—

Tables of Mortality.

Ratio of increase of the Births.		1	)0	Apparent. 1.005	1.010	1.015	1.020
Births.	10000	2549	)0	10000	10000	10000	10000
Between	Living.		7	6294	5695	5115	4551
0 and 5	40512	10320		5914	5265	4638	4037
5 — 10	33096	8430		5515	4826	4160	3550
10 — 20	63083	16082		5050	4345	3680	3066
20 — 30	58980	15035		4492	3794	3146	2563
30 — 40	53894	13747		3855	3197	2596	2067
40 — 50	47642	12143		3184	2596	2068	1619
50 — 60	40955	10441		2129	1695	1315	1004
60 — 70	30826	7853		864	667	503	370
70 — 80	17502	4462		132	99	71	50
80 — 90	5188	1329					
90 — 100	512	13					
	392190	10000					
			48	36.193	31.568	29.283	23.451

	By Mr. Finlaison's Table Deaths in a Population of	
	10000 Males.	10000 Females.
0		
5	10.59	7.06
10	6.78	5.98
15	9.40	8.94
20	19.32	13.41
30	13.74	11.15
40	12.44	10.05
50	16.17	9.45
60	18.92	11.55
70	18.49	15.49
80	10.08	10.81
90	1.00	4.60
100		

TABLE

	America.	London Bills of Mortality 1629—1636.
Asthma...........	.002	.000
Consumption......	.168	.194
Convulsions......	.048	.039
Dropsy...........	.060	.041
Fever.............	.115	.103
Inflammation......	.046	.000
Measles...........	.012	.003
Small Pox.........	.000	.046
Other Diseases....	.546	.569

Births.									
	Florence,* 1451 to 1734.	Paris,* 1670 to 1787.	Sweden.	Montpellier.	London.	Brussels.*	Amsterdam.*	Netherlands.*
8	1.12	1.00	1.00	1.13	.90	1.04	1.05	1.08
5	1.19	1.39	1.09	1.09	1.01	1.15	1.10	1.18
5	1.13	.23	1.00	.98	.97	1.09	1.10	1.17
0	1.01	.71	1.08	.88	.96	1.07	1.50	1.08
2	.90	.95	1.02	.86	.92	.98	1.01	1.96
8	.83	.82	1.10	.85	.97	.95	.98	.86
2	.85	.87	1.01	.92	.91	.90	.93	.82
4	.92	.80	.91	1.00	.89	.90	.89	.89
9	.95	.85	.89	1.01	.88	.94	.91	.97
9	1.03	.89	.85	1.05	.81	.94	.94	.98
3	1.07	1.20	.95	1.16	1.09	.96	.97	.99
4	.94	.13	1.10	1.07	1.69	1.17	1.01	.97

cs, Recherches sur la Population, &c.

an individual died in the year in which he is recorded to have died, e_1 the year after, e_n the n^{th} year after, &c., and if the Table of mortality be founded upon a population observed from birth throughout life, upon the same hypothesis of probability *a priori* as before, the formula which I gave in my former Paper on this subject, p. 152, shews that if d_n be the number of deaths recorded to have taken place at the n^{th} age, the probability at the birth of a child, that he will die at the n^{th} age is

$$\frac{\Sigma\{d_{n+m} \times e_m\}+1}{\Sigma d_n + p}.$$

Σd_n being the total number of persons observed, and p the number of cases possible, or ages at which deaths are supposed to take place. The values of e are to a certain extent arbitrary. If e be supposed to be constant and $=\frac{1}{m+1}$, and that the values of e are $e_{-\frac{m}{2}} \ldots e_{-1}$, e, $e_1 \ldots e_{\frac{m}{2}}$, this amounts to taking the mean of the deaths which are recorded to have taken place within $\frac{m}{2}$ years of the age n. Generally, however e be supposed to vary, $\Sigma e_m = 1$. This theory shews how a Table of mortality should be corrected for the irregularities which present themselves, when the observations are not numerous.

The number p may also be considered as arbitrary, and by altering this which amounts to increasing the deaths at every age by an arbitrary quantity, the Table may also be corrected, but the former method is simpler.

With the assistance of Mr. Deacon I have calculated the Tables of Annuities, at the end of this Paper, by the approximate method given above, and the data or table of observations from which they are taken is prefixed to each.

3. Table (1) contains different registers of mortality, giving first the actual number of living deduced from the recorded deaths, and then the same reduced to the radix 1000.

The Table for Paris is taken from the *Annuaire du Bureau des Longitudes.*

The Breslaw Table is taken from Dr. Halley's Paper in the Transactions of the Royal Society, it was formed from observations communicated to the Royal Society by Mr. Justell. Dr. Halley has not given the observations themselves.

Kerseboom's Table was formed by him from registers of Life Annuitants in Holland and West Friesland. Desparcieux's Tables from lists of the nominees in the French Tontines, these two must be considered as formed upon very select life.

The Tables for Brussels and Amsterdam are taken from the *Recherches sur la Population dans le Royaume des Pays Bas,* by Mr. Quetelet.

The Table for Sweden was formed "from observations of the proportion of the living to the numbers who died at all ages for 21 years, from 1755 to 1776, in the kingdom of Sweden." See Dr. Price, Vol. II. p. 140. The Table for Montpellier is from a Memoir by Mr. Morgue, in the first Vol. of the *Memoires de l'Institut.* The Northampton Table is taken from the deaths in All Saints' Parish, Northampton, from 1735 to 1780. See Dr. Price, Vol. II. p. 95. The Carlisle Table of mortality as given by Mr. Milne, was formed by him from the observations of the mortality which are given in the next column, combined with two enumerations of the population. The numbers upon which this Table is formed are very small. The expectation of life is given at the foot, calculated from each by a method similar to that I have explained for calculating annuities.

Table II. contains annuities deduced from the preceding.

Table III. contains Tables of mortality in which the sexes are distinguished, and Table IV. contains annuities deduced from them: it will be observed, that all these Tables agree in giving to females a greater longevity than to males; a fact, which is further confirmed by the circumstance that in all countries, with the exception, I believe, of Russia, notwithstanding the male births exceed the female, the number of females in the population exceeds that of the males.

Mr. Griffith Davies has published Tables of Annuities taken from statements of Mr. Morgan in his addresses to the general Courts of the Equitable Society, and in notes added by him to the latter editions of Dr. Price's Observations on Reversionary Payments. In Mr. Morgan's address to the general Court held on the 24th of April 1800, he stated that the decrements of life among the members of the Equitable for the preceding 30 years had been to those of the Northampton,

From	10 to 20	as	1 : 2
	20 ... 30	...	1 : 2
	30 ... 40	...	3 : 5
	40 ... 50	...	3 : 5
	50 ... 60	...	5 : 7
	60 ... 80	...	4 : 5

which statement is confirmed in his subsequent addresses.

In a recent publication Mr. Morgan admits that he was not then aware of the great number of instances in which there are several policies on one and the same life, and he says that this circumstance very materially affects Mr. Davies's calculations.

Such statements as these appear to me too vague to be made the basis of calculations, although the experience of the

Equitable Society would be most valuable, if we were acquainted with all the details concerning it.

Mr. Finlaison has very recently published extensive Tables of mortality formed from the Government Tontines and Annuitants, which are rendered equally valuable by the accuracy of the materials from which they have been deduced, and the very great care and attention which has been bestowed on them by the author. Mr. Finlaison has done me the favor to prepare for me a summary of these Tables, which is to be found in Table V. in a form in which it may be easily compared with the other Tables which I have given.

Mr. Finlaison (in his report to the Lords of the Treasury) explains at length the manner in which he made use of the records of the Tontines. Mr. Finlaison observes "that the facts shown in these observations bear conclusive testimony that the rate of mortality in England has, during the last century, diminished in a very important degree, on each sex equally, but not by equal gradations, nor equally at all periods of life; and that while in regard to the males it seems in early and middling life to have remained for a long time as it stood about fifty years ago; in respect of the females it has during the same time visibly and progressively diminished to this day by slight but still sensible gradations." This fact is at variance with the opinion that the improvement which has taken place in life is to be attributed to the introduction of vaccination. Epidemics however are of much less frequent occurrence in England than they were formerly, which circumstance must tend materially to diminish the rate of mortality.

The great plague years in London were 1592, 1593, 1603, 1625, 1636 and 1665, in which the burials were as follows:

A.D.	1592.	1593.	1603.	1625.	1636.	1665.
Total Deaths.	25886	17844	37294	51758	23359	97306
Deaths of the Plague.	11503	10662	30561	35417	10460	68596

Now the average number of deaths in London is about 20,000, and the actual number varies very little.

Observations such as those presented by Mr. Finlaison, where the deaths are given at every age, are particularly well calculated to determine delicate points, such as any small increase of the rate of mortality at different ages. A small increase of mortality according to Mr. Finlaison's Tables takes place about 23, thus in observation 19, p. 56, of Mr. Finlaison's report, it appears that there is a minimum of mortality at 13, a maximum at 23, and a minimum again at 33. This does not obtain in Mr. Finlaison's observations on females. It is very remarkable that the same circumstance is to be observed in the Chester Tables, though here it is found equally in the Tables for males and females: this appears to me a great proof of their accuracy, and of the fidelity with which Dr. Haygarth recorded the facts which were presented to him. Dr. Price says, "The Bills (for Northampton) give the numbers dying annually between 20 and 30 greater than between 30 and 40, but this being a circumstance which does not exist in any other register of mortality, and undoubtedly owing to some accident and local causes, *the decrements were made equal between* 20 *and* 40," &c. Vol. II. p. 97.

However accurate the observations be upon which Mr. Finlaison's results are founded, it must be recollected that the lives were selected from a selected class, and it remains to be shewn, that the mortality in the lower classes of society is the same as in the higher, and that selection produces no effect on the results.

4. Tables of mortality which are founded upon registers of deaths only are subject to an error arising from the supposition that the population is stationary, as was long ago noticed by Dr. Price, Vol. II. p. 251.

The probability of an individual dying in a given n^{th} year of his life, if the effect of migration be neglected, is the number of deaths of that age divided by the number of births in one year, n years previously, which, if the population were stationary, would be the same as the total of deaths in any year.

If therefore the births n years previously are $\gtrless$ than the total of deaths at all ages in the year of the observation, the probability of an individual dying at the n^{th} age is $\lessgtr$ than the quotient of the deaths at that age divided by the total of the deaths at all ages. In America this effect is I think clearly perceptible, and has led some persons to conclude that the population in that continent is more unhealthy than in Europe.

The following Table has been formed from the Bills of Mortality for Boston, New York, Philadelphia and Baltimore in 1820

Age.	Living.	Age.	Living.
0	1000	40	254
5	587	50	160
10	549	60	96
20	495	70	53
30	371	80	24

Expectation of Life at Birth 24.959,

which Table is much lower than any of the others, but the annual rate of increase of the population in the United States between 1810 and 1820 was about 1.034. In England at the

same time it was only 1.016. In order to shew directly the effect which an increase in the population produces in the Table of mortality, I have calculated three Tables from the Chester Tables of mortality, supposing the deaths at the time of the observation to be equal to the deaths 40 years previously (which was nearly the case in this country in the last century), and the births to increase annually in a geometrical progression of which the common ratio is given.

The column *A* supposes the ratio of increase to be 1.005,

.... *B* 1.010,

.... *C* .... 1.015.

The column *D* is calculated in the same way for females, and supposes the ratio of increase to be 1.005. The ratio 1.005 is very nearly what obtained in England during the last century according to the Parliamentary Reports. The births in all England in the year 1700 were 138.979, and in 1780, 201.310, making the mean annual rate of increase 1.0046: in the county of Chester taken by itself in 1700 they were 2690, and in 1780 4592, making the mean annual rate of increase 1.0061; therefore the columns *A* and *D*, which I have given at length in Table VII. must approach very nearly to exactitude, and considering attentively the limits of the errors of which observations of this kind are susceptible, I think that it is improbable that the longevity in this country generally, when the Chester Table was formed, was quite so great as that indicated by Mr. Finlaison's Tables and the experience of the Equitable Society. It may have improved since.

When the law of mortality in any country, and the number of births in each year during the century previous to any given epoch, are known, it is easy to assign the total number of

persons living at every age. For if $p_{0,n}$ be the probability of a child at birth surviving n years, b_n the births n years previously, the number of living in the population at the n^{th} age is $p_{0,n} \times b_n$, and the ratio of the living at that age to the whole population is $\frac{p_{0,n} \times b_n}{\Sigma (p_{0,n} \times b_n)}$.

I have calculated Tables VIII. and IX. in order to shew the effect which is produced by a given increase of the births. Table VIII. shews the proportion of the living at each age, and of the deaths to the whole population, when the law of mortality obtains which is given by Table VII. The male births are supposed to be to the female as 104 to 100. Table IX. is calculated upon the supposition, that the law of mortality obtains which is given by the Carlisle Table in Mr. Milne's work, Vol. II. p. 564. The following are the results which are given by these Tables:

Ratio of Increase of the Births yearly...	1		1.005		1.010		1.015		1.020	
	Chester.	Carlisle.	Chester.	Carlisle.	Chester.	Carlisle.	Chester.	Carlisle.	Chester.	Carlisle.
Ratio of the Births to the Population......	$\frac{1}{37.381}$	$\frac{1}{39.219}$	$\frac{1}{32.092}$	$\frac{1}{33.783}$	$\frac{1}{27.964}$	$\frac{1}{29.274}$	$\frac{1}{24.660}$	$\frac{1}{25.654}$	$\frac{1}{21.930}$	$\frac{1}{22.722}$
Ratio of the Deaths to the Population......	$\frac{1}{37.381}$	$\frac{1}{39.219}$	$\frac{1}{38.417}$	$\frac{1}{39.440}$	$\frac{1}{38.409}$	$\frac{1}{40.054}$	$\frac{1}{38.544}$	$\frac{1}{40.085}$	$\frac{1}{38.187}$	$\frac{1}{39.781}$
Ratio of Increase of the Population yearly...			1.005	1.005	1.010	1.010	1.015	1.015	1.020	1.020
Population doubles in...			138 years	138 years	69 years	69 years	46 years	46 years	35 years	35 years
Deaths are equal to the Births after....			36 years	31 years	33 years	31 years	30 years	30 years	27 years	28 years

The ratio of the deaths to the population is nearly constant according to both these Tables, whatever be the rate of increase of the births; when the ratio of the births to the population

is constant, the rate of increase of the population is necessarily the same as that of the births. The rate of increase of the births has been supposed to be constant; a small inequality in this rate, unless it be of long period, will not produce any sensible difference in the results. But, although the total number of deaths which take place in a given population is not much influenced by the rate of increase, the apparent table of mortality is much altered. In order to shew the extent of the error which is likely to arise from this circumstance, I have given the apparent tables of mortality corresponding to each rate of increase of the births.

According to Mr. Rickman, in the Population Abstract 1821, the ratio of the deaths to the population in England at that time was 1 to 57. This ratio is considerably less than would be given by any table of mortality, and it is probable, therefore, that the number of unentered burials is much greater than Mr. Rickman has supposed. Since the ratio of the deaths to the population is nearly constant when the law of mortality is given, this rate would be an excellent criterion of the longevity of different countries, if it could be accurately ascertained; to this, however, many difficulties are opposed.

In the Tables VIII. and IX. the rate of increase of the births is arbitrary; in order to see how far the mortality in this country coincides with that given by Table VII., I have formed Table X., taking the values of p from that Table, and supposing the births in the century previous to 1821 to have been the same as the christenings that are given in the Population Abstract before referred to; and since the ratio $\frac{p_{0,n} \times b_n}{\Sigma(p_{0,n} \times b_n)}$ involves only the ratios of the births, which must be nearly the same as the ratios of the christenings, the error introduced by this hypothesis is altogether insensible.

I have placed, for the sake of comparison, the results given by the census of 1821 with the results deduced from theory, and they agree, I think, within the limits of the errors of which the census is susceptible, and much nearer than the results of different counties agree with each other. The number of deaths in a population of 1000 males and females, according to the law of mortality of Table VII. is 271, making the ratio of the deaths to the population about $\frac{1}{37}$. Calculating the deaths between 0 and 5, to which period Mr. Finlaison's Observations do not extend, from the same Table, and those at the succeeding ages from Mr. Finlaison's Observations, 11 and 19, the total number of deaths which results in a population of 1000 males and females is 244, nearly; and the ratio of the deaths to the population about 1 to 41: which is far greater than the ratio given by Mr. Rickman.

The following are some of the elements of the population of England and France. Those for England are deduced from the returns in the Population Abstract of 1821, before referred to, and those for France from the *Annuaire du Bureau des Longitudes* for 1829.

	England.	France.
Ratio of males to females..................	.95764 : 1	
——— male births to female...............	1.0435 : 1	1.0656 : 1
————deaths to female............	1.0024 : 1	1.0180 : 1
————legitimate births to female....		1.06795 : 1
————illegitimate births to female...		1.04844 : 1
——— population to marriages in one year.	122.50 : 1	132.619 : 1
———— births in one year...	32.274 : 1	31.535 : 1
———— deaths in one year....	54.296 : 1	39.423 : 1
——— births to marriages................	3.5902 : 1	4.205 : 1
——— legitimate births to marriages......		3.912 : 1
——— increase of the population annually.	1.0167	1.00634

The population of England according to the census of 1811 was 9,538,827, and according to that of 1821, 11,261,437, making the mean annual rate of increase of the population 1,0167.

The baptisms in 1810 were 298,853, and in 1820, 343,660, making the mean annual rate of increase 1,0140.

Mr. Richman considers the census of 1821 more accurate than that of 1811, if therefore we suppose the ratio of the births to the population to have been constant during this short interval between these enumerations, so that the real rate of increase of the population was only 1,0140, we have 9,792,600 for the population in 1811 instead of 9,538,827, and 1,468,837 for the increase of the population between 1811 and 1821. A comparison of the registered baptisms and burials during the same time gives an apparent increase of only 1,245,000. See Mr. Richman's observations prefixed to the Population Report 1821.

Hence if the increase was really 1,468,837, the average yearly excess of unentered baptisms over unentered burials is 22,383, and if with Mr. Richman we admit the average number of unentered burials yearly to be 8,770, the average number of unentered baptisms will be 31,153. The baptisms in England in 1820, were 328,230.

$$\frac{328,230 + 22,383}{11,261,437} = \frac{350613}{11,261,437} = \frac{1}{32,044},$$

which ratio does not materially differ from that given above, in deducing which the average yearly number of unentered baptisms was supposed to be 20,696. The ratio of the population to the deaths was found by adding 8,770 to 198,634 the total of the burials in 1820, and to the marriages by adding 191 to 91,729 the marriages in the same year and dividing by 11,261,437. See p. 145 of the Report above alluded to.

Mr. Benoiston de Chateauneuf in the *Annales des Sciences Naturelles* 1826, gives the following numbers as the ratio of the births to the marriages in

Portugal 5.14.	Bohemia 5.27.
Lombardy 5.45.	Muscovy 5.25.

and in several of the southern departments of France above 5.

In the territory of the two Sicilies the ratio in 1828 according to the Report of the Secretary of State was 5.716 : 1.

This ratio is increased by two causes, either by the prolificness of the sex, or by the prevalence of concubinage. In the report above alluded to, the ratio of the marriages to the population is 1 : 154, in England it is 1 : 122, which difference is sufficient to account for the difference in the ratio of the births to the marriages without supposing the former of the two causes indicated above to exist.

If the ages at which deaths take place, and the number of births were accurately registered in a great empire, the probabilities of life would be known with the greatest accuracy, the multitude of the observations destroying any small sources of inaccuracy, and the number of the population $(= \Sigma p_{0,n} \times b_n)$ would be known far more accurately than by the laborious process of actual enumeration, for in a large district the effect of migration would be wholly insensible. It seems indeed worthy of consideration whether it might not be possible to publish annually the Bills of Mortality for every parish in the empire, as is now done in London and in some great towns. If this were done, many interesting questions in science would be determined, the comparative healthiness of different districts and of different periods of the year would be ascertained, and great light might be thrown upon the efficacy of the manner in which different diseases are treated. So many questions in which property is involved, are connected with the accuracy of the parish books, that it seems extraordinary that greater attention is not paid to their exactness.

No data have yet been published by which the additional premium can be determined, which should be paid when the subject of the policy has any chronic disease. The only case of which I have endeavoured to determine the risk is child-birth. The deaths in child-birth during the ten years from 1818 to 1827 by the London bills were 2117, the number of christenings 241352, and the number of still-born 7575, which would give $\frac{2117}{248927}$ or $\frac{1}{117}$, for the probability that a woman does not survive giving birth to a child, making the extra premium of insurance about 17*s*. At Strasburg the deaths in childbirth are 1 in 109. At the City of London Lying-in Hospital in 1826, the deaths were 1 in 70; in the Dublin Hospital in 1822, there were 12 deaths among 2675 women delivered, or 1 in 223; in the Edinburgh Hospital the mortality is 1 in 100; in the whole kingdom of Prussia in 1817, the deaths were 1 in 112. See Dr. Hawkins's Medical Statistics. Most extensive returns of sickness have been furnished to the Society for the Diffusion of Useful Knowledge by Friendly Societies, and these will no doubt furnish much valuable information upon the subject of the duration of sickness. If returns could be obtained from Hospitals of the ages at which individuals come in afflicted with different complaints, with the time they continue under treatment, and the number who die, these would also furnish the means of determining the probability of a sick person continuing sick for any given time, and the probability of an individual sick dying; from this and the probability of an individual dying at the given age which is given by the Tables, the probability of an individual falling sick at a given age with his *expectation* of sickness at that age might be determined. The Bills of Mortality in London give the diseases by which deaths are occasioned, but unfortunately the sexes are not distinguished.

Table IX. shews the ratios of the diseases to which the deaths have been attributed at different periods in the London Bills. Measles seem to have increased; so little dependence, however, is to be placed on these documents, that I forbear making any further comments upon them. The column headed America is taken from the Bills of Mortality for Boston, New York, Philadelphia and Baltimore, and that for Carlisle from Mr. Milne's work on Annuities.

I have also endeavoured to determine from the Bills of Mortality, as given in the Annual Register for the ten years from 1810 to 1820, the mortality and the births in London at different seasons, see Table XII. The burials amounted during this period to 197,695, and the christenings to 245,287.

The returns however are made so very irregularly, that these results notwithstanding the very large numbers from which they are formed are by no means accurate, for the parish clerks, as I find by examining the Weekly Bills, generally return the deaths and christenings of several weeks together. I have annexed observations of a similar kind given by Mr. Quetelet and Mr. Milne, and a Table for Glasgow, which I have deduced from the Bills of Mortality for that city for the years 1821 to 1827, the total number of burials during that time was 31,245.

In London the mean monthly price of wheat varies very little, if at all, the same is the case with the barometer; the variation therefore which takes place in the number of deaths and christenings, must be principally owing to the variations in the temperature. The mean number of christenings in any month, in a given place, will also be affected by the mean time which christening is delayed after birth in that place. All the results given in Table XII. have been reduced to the radix 1200, and are corrected for the unequal lengths of the months.

I have thus endeavoured, as briefly as possible, to present the data which we now possess for determining questions connected with the duration of human life. The accordance of the results which have been deduced, proves that no considerable error can obtain, for the slight difference which exists between Table VII. which I have formed from the observations at Chester, and the Table formed by Mr. Milne from those at Carlisle, is of the order of the inevitable errors of these observations, and of the hypothesis I made with respect to the rate of increase of the population during the century previous to the observation; and in order to get rid entirely of this slight discrepancy, it would be only necessary to make the rate of increase about 1.007 instead of 1.005 as I supposed it to be. The Northampton Table treated in the same way would give results nearly similar.

No doubt our information on this subject will soon be much improved; for when we consider the accuracy which has been introduced into every other branch of philosophical enquiry, it appears surprising that this should have remained so far behind.

J. W. LUBBOCK.

A

LIST OF DONATIONS

TO THE

LIBRARY AND MUSEUM

OF THE

CAMBRIDGE PHILOSOPHICAL SOCIETY,

SINCE FEB. 26, 1827.

I. *Donations to the* LIBRARY.

1827.		DONORS.
April 20.	VITA e commercio literario di Galileo	Rev. D. Pettiward.
	Memorie e lettere di Galileo Galilei	————
May 14.	Account of Observations with a twenty feet reflecting telescope.................	J.F.W. Herschel, Esq.
	The Elements of Plane Trigonometry .	Rev. J. Hind.
	Observations on the Effect of Tithes upon Rent	J. Buckle, Esq.
Nov. 12.	Philosophical Transactions, 1826, Part 4. and 1827, Part 1........................	
	Geological Transactions, Vol. II. Parts 1. and 2....................................	Geological Society.
	Transactions of the Linnean Society, Vol. XIV. and Vol. XV. Part 1.....	Linnean Society.
	Transactions of the American Society, 5 Vols. of the old series, and 1 Vol. of the new................................	American Society.
	Memoires de la Societé de Physique et d'Histoire Naturelle de Geneve, 4 Parts	Soc. de Geneve.
	Eulogium on Thomas Jefferson, Esq....	

		Donors.
1827.		
	Letter to Mr. Canning	Dr. Wade.
	Notes in Defence of the Colonies	
Dec. 10.	Transactions of the American Society, Vol. II.	American Society.
	Manual of Electro-Dynamics	Prof. Cumming.
	Nat. Hist. of Stockton on Tees	J. Hogg, Esq.
	On Systems and Methods in Natural History	J. E. Bicheno, Esq.
1828.		
Mar. 17.	Niebuhr's History of Rome	Rev. J. C. Hare and Rev. C. Thirlwall.
April 21.	Article Light, Encyc. Metrop.	J.F.W. Herschel, Esq.
	On the Geometrical representation of impossible Quantities	Rev. J. Warren.
	Armenian Dictionary	Rev. D. Pettiward.
	Armenian Grammar	J. Hogg, Esq.
	On Public and Church Clocks	B. L. Vulliamy.
	On Conducting Private Bills in the House of Commons, by T. M. Sherwood	T. Thorp, Esq.
	Daubeny on Volcanoes	Prof. Sedgwick.
May 5.	Memoire sur les Belemnites	M. Blainville.
	Analyse des Travaux de l'Acad. Royale pendant l'année 1826. Partie Mathematique	J. Underwood, Esq.
	Discours prononcés aux Funerailles de M. de la Place	————
	Ceremonies of the University	H. Gunning, Esq.
May 19.	On the Osteology of Chlamyphorus Truncatus	W. Yarrell, Esq.
	Observations on the Tracheæ of Birds	————
	On the Plumage of some Hen-Pheasants	————
Nov. 11.	On the Curative influence of the Sea Coast	Dr. Harwood.
	Outlines of a Penal Code	T. Disney, Esq.

		Donors.
1828.		
	Observations at Port Bowen	Lieut. Foster.
	Transactions of the Royal Society, 1828. Part 1....................................	
	Third Catalogue of Double Stars.......	J.F.W. Herschel, Esq.
	Supplement to Practical Astronomy....	Dr. Pearson.
1829.		
Mar. 2.	Further Remarks on the Nautical Almanack....................................	F. Bailey, Esq.
	On the Occultation of δ Piscium.......	J. South, Esq.
	Practical Astronomy, Vol. II.	Dr. Pearson.
	Edinburgh Transactions, Vol. XI. P. 1.	Edinburgh Society.
	Asiatic Transactions, Vol. II. P. 1.....	Asiatic Society.
Mar. 16.	Kirby and Spence's Entomology, 5th Edition....................................	Rev. W. Kirby.
	Vindication of Niebuhr....................	Rev. J. C. Hare.
Mar. 30.	On the Promotion of Astronomical Science....................................	J. South, Esq.
	Report of the House of Commons on the Board of Longitude and on the Nautical Almanack.....................	Rev. R. Sheepshanks.
May 18.	Pryce's Mineralogia Cornubiensis	J. Carne, Esq.
	Cambridge Observations....................	Prof. Airy.
	Maclurean Museum of Arts and Sciences, Nos 1. and 2.....................	Maclurean Society.
	Sur les Continens Actuels..................	M. Constant Prevost.
	* Audubon's Coloured Plates of North American Birds..........................	
	* Mrs. Bowdich's Drawings and Descriptions of British Freshwater Fish.	

*** These splendid works have been purchased by joint Subscriptions of several of the Members, who considered them desirable acquisitions to the Society's Library. Some Members may require to be informed that the Society having no funds which can be applied to such a use, this is the only way in which works of this kind can be procured. It is proposed to continue them in the same manner by the assistance of such Members as may be disposed to contribute.

II. *Donations to the* Museum.

		Donors.
1827.		
Dec. 5.	Sword of the Sword-Fish	Dr. Thackeray.
1828.		
Mar. 17.	Caryophylla Ramea.........................	Rev. R. T. Lowe.
April 21.	Barometer, by Troughton	E. Troughton, Esq.
	Thermometer, do............................	———
	Roman Urn from Trumpington	Dr. Thackeray.
Nov. 11.	Concretions from the Stomach of a Horse..	Rev. R. Lynn.
	Saw of a Saw-Fish	Capt. Mortlock.
	Hercules Beetle	———
1829.		
Nov. 16.	Stuffed Specimen of the Hen Harrier...	Geo. Jenyns, Esq.

A Collection of British Birds has been purchased by a voluntary Subscription of the Members of the Society, and is placed in Cases surrounding the Reading Room. The following is a list of the Specimens.

COLLECTION OF BRITISH BIRDS,

PURCHASED BY

THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

N. B.—The asterisk distinguishes those which are not British killed specimens.
The figures refer to the large, and the letters to the smaller Cases.

Case I

1. Haliæetus Albicilla. *White-tailed Eagle. Adult.*
2. ———— *Young.*
*3. Aquila Chrysäetos. *Golden Eagle. Young.*
4. Pandion Haliætus. *Osprey.*

Case II.

*1. Aquila Chrysäetos. *Golden Eagle. Adult.*
2. Accipiter Fringillarius. *Sparrow Hawk. Adult male.*
3, 4. ———— *Adult females.*
5. Falco Tinnunculus. *Kestril. Adult male.*
6. ———— *Adult female.*
7. —— Æsalon. *Merlin. Adult male.*
8. ———— *Nearly adult male.*
9. ———— *Young mal.*
10. ———— *Adult male.*
11, 12. ———— *Young males.*
*13. ———— *Adult female.*
14. ———— *Young male.*

Case III.

1. Milvus Ictinus. *Kite. Adult female.*
2. Astur Palumbarius. *Goshawk. Adult male.*
3. Milvus Ictinus. *Kite. Adult male.*
4. Falco Subbuteo. *Hobby. Adult female.*
5. ———— *Adult male.*
*6. Astur Palumbarius. *Goshawk. Young male.*
7. Falco Islandicus. *Jer-Falcon. Adult male.*
8. —— peregrinus. *Peregrine Falcon. Adult female.*
9. ———— *Adult male.*
10. ———— *Young male.*

Case IV.

1. Buteo Lagopus. *Rough-legged Buzzard. Adult male.*
2. ———— *Adult female.*
3. Pernis apivorus. *Honey Buzzard. Young male.*
4. ———— *Adult male.*
5. Circus æruginosus. *Moor Harrier.*
6. —— Pygargus. *Hen Harrier. Adult male.*
7. ———— *Ringtail. Adult female.*
8. Buteo vulgaris. *Common Buzzard.*
9. Circus Pygargus. *Ringtail. Young.*

Case V.

*Bubo maximus. *Eagle Owl. Male.*
*———— *Female.*
Otus vulgaris. *Long-eared Owl.*
—— brachyotus. *Short-eared Owl.*
*Scops Aldrovandi. *Scops-eared Owl.*
*Noctua nyctea. *Snowy Owl. Male.*
*———— *Female.*
—— passerina. *Little Owl.*
Strix flammea. *White Owl.*
Syrnium Aluco. *Tawny Owl.*

Case A.

1 & 2. Lanius Excubitor. *Ash-coloured Shrike.*
3. —— Collurio. *Red-backed Shrike. Male.*
4. ———— *Female.*
*5 & 6. —— rufus. *Woodchat.*

Case VI.

1. Corvus frugilegus. *Rook.*
2. —— Corax. *Raven.*

3. Corvus Monedula. *Jackdaw.*
4. ——— Corone. *Carrion Crow.*
5. Fregilus europæus. *Red-legged Crow.*
6. Corvus Cornix. *Hooded Crow.*
7 & 8. Garrulus glandarius. *Jay.*
*9. Nucifraga Caryoctatactes. *Nutcracker.*
*10. Coracias garrula. *Roller.*
11 & 12. Pica europæa. *Magpie.*
13 & 14. Turdus viscivorus. *Missel Thrush.*
15 & 16. ——— Pilaris. *Fieldfare.*
17 & 18. ——— iliacus. *Redwing.*
19. ——— musicus. *Thrush.*
20. ——— torquatus. *Ring Ouzel. Female.*
21. ———————————— *Male.*
22. ——— Merula. *Blackbird. Male.*
23. ———————————— *Female.*
*24. Oriolus Galbula. *Golden Oriole. Male.*
*25. ———————————— *Female.*
*25 & 26. Bombycivora garrula. *Waxen Chatterer.*
*27. Pastor roseus. *Rose-coloured Ouzel.*
28 & 29. Sturnus vulgaris. *Starling.*

Case VII.

1. Muscicapa atracapilla. *Pied Flycatcher.*
2. ——— grisola. *Spotted Flycatcher. Adult.*
3. ———————————— *Young.*
4 & 5. Curruca salicaria. *Sedge Warbler.*
6 & 7. ——— Locustella. *Grasshopper Warbler.*
8. ——— arundinacea. *Reed Warbler.*
9 & 10. ——— provincialis. *Dartford Warbler.*
11 & 12. ——— Luscinia. *Nightingale.*
13. ——— sylvia. *Whitethroat. Male.*
14. ———————————— *Female.*
15—17. ——— sylviella. *Lesser Whitethroat.*
18 & 19. ——— atracapilla. *Blackcap. Male.*
20 & 21. ———————————— *Female.*
22 & 23. Sylvia Rubecula. *Robin.*
24 & 25. Curruca hortensis. *Pettychaps.*
26 & 27. Regulus Trochilus. *Willow Wren.*
28. Curruca Sibilatrix. *Wood Wren.*
29 & 30. Regulus Hippolais. *Lesser Pettychaps. Adult?*
31 & 32. ———————————— *Young?*
33 & 34. Troglodytes europæus. *Common Wren.*
35 & 36. Regulus cristatus. *Golden-crested Wren. Male.*
37. ———————————— *Female.*
38. Saxicola Rubetra. *Whinchat. Male.*
39. ———————————— *Female.*
40. ——— rubicola. *Stonechat. Male.*
41. ———————————— *Female.*
42. Sylvia Phœnicurus. *Redstart. Male.*
43. ———————————— *Female.*
44. Saxicola Œnanthe. *Wheatear. Male.*
45. ———————————— *Female.*
*46. Accentor alpinus. *Alpine Warbler.*
47 & 48. ——— modularis. *Hedge Warbler.*
49. Motacilla alba. *White Wagtail. Winter plumage.*
50. ———————————— *Summer plumage.*
51. ——— flava. *Yellow Wagtail. Male.*
52. ———————————— *Female.*
53. Boarula. *Grey Wagtail. Summer plumage.*
54. ———————————— *Winter plumage.*
55. Alauda arvensis. *Skylark.*
56 & 57. ——— arborea. *Woodlark.*
58. Anthus arboreus. *Tree Pipit.*
59—61. ——— petrosus. *Rock Pipit.*
62 & 63. ——— pratensis. *Tit Pipit.*

Case VIII.

1. Parus major. *Great Titmouse. Female.*
2. ———————————— *Male.*
3 & 4. —— palustris. *Marsh Titmouse.*
5 & 6. —— ater. *Cole Titmouse.*
7—9. —— caudatus. *Long-tailed Titmouse.*
10. —— cristatus. *Crested Titmouse.*
11 & 12. Parus cœruleus. *Blue Titmouse.*
13. —— biarmicus. *Bearded Titmouse. Male.*
14. ———————————— *Female.*
15. ———————————— *Young.*
16 & 17. Emberiza Miliaria. *Common Bunting.*
18. ——— Citrinella. *Yellow Bunting. Female.*
19. ———————————— *Male.*
20. ——— Cirlus. *Cirl Bunting. Male.*
21. ———————————— *Young.*
22. ———————————— *Female.*
*23. ——— Hortulana. *Ortolan Bunting.*
24. ——— schœniclus. *Reed Bunting. Male.*
25. ———————————— *Female.*
26. ———————————— *Male.*
27. Plectrophanes nivalis. *Snow Bunting. Adult.*
28 & 29. ————————————
*30. Coccothraustes vulgaris. *Hawfinch. Female.*
*31. ———————————— *Male.*
32. ——— Chloris. *Greenfinch. Male.*
33. ———————————— *Female.*

34. Pyrgita domestica. *House Sparrow. Male.*
35. ———————— *Female.*
36 & 37. ——— montana. *Tree Sparrow. Male.*
38. Fringilla cœlebs. *Chaffinch. Male.*
39. ———————— *Female.*
40—42. ——— Carduelis. *Goldfinch.*
43. ——— Montifringilla. *Brambling. Male.*
44. ———————— *Female.*
45. ——— cannabina. *Greater Redpole. Female.*
46. ———————— *Male.*
47. ————————
Young.
48. ——— Linaria. *Lesser Redpole. Male.*
49. ———————— *Young.*
50. ———————— *Female.*
51. ——— Spinus. *Siskin. Male.*
52. ———————— *Female.*
53 ——— Linota. *Linnet. Female.*
54. ———————— *Male.*
55 & 56. ——— Montium. *Twite.*
57. Pyrrhula vulgaris. *Bullfinch. Female.*
58. ———————— *Male.*
*59. Corythus Enucleator. *Pine Grosbeak. Male.*
*60. ————————
Female.
61. Loxia curvirostra. *Crossbill.*

CASE B.

*1. Picus martius. *Great Black Woodpecker. Male.*
2. ————————
Female.
3. —— viridis. *Green Woodpecker. Male.*
4. ———————— *Female.*
5. —— major. *Greater spotted Woodpecker. Male.*
6. ————————
Female.
7. —— minor. *Lesser spotted Woodpecker. Male.*
8. ————————
Female.

CASE C.

1. Yunx Torquilla. *Wryneck. Male.*
2. ———————— *Female.*
3 & 4. Certhia familiaris. *Creeper.*
5 & 6. Sitta europœa. *Nuthatch.*
7 & 8. Cuculus canorus. *Cuckow.*
9. ———————— *Young.*

CASE D.

1. Upupa Epops. *Hoopoe.*
*2 & 3. Merops Apiaster. *Bee-eater.*
4 & 5. Alcedo Ispida. *Kingsfisher.*

CASE E.

*Cinclus aquaticus. *Water Ouzel.*

CASE IX.

*Tetrao Urogallus. *Wood Grouse. Male.*
*———————— *Female.*
—— Tetrix. *Black Grouse. Male.*
———————— *Female.*
Lagopus scoticus. *Red Grouse. Male.*
———————— *Female and Young.*
——— mutus. *Ptarmigan. Winter plumage.*
———————— *Summer plumage.*

CASE X.

*1. Otis tarda. *Great Bustard. Male.*
2. ———————— *Female.*
*3. —— Tetrax. *Little Bustard. Male.*
*4. ———————— *Female.*
5. Œdicnemus crepitans. *Great Plover. Male?*
6. ———————— *Female?*

CASE XI.

*1. Ardea Garzetta. *Egret.*
*2. Grus cinerea. *Crane. Female?*
*3. ———————— *Male.*
*4. Ardea Nycticorax. *Night Heron. Adult.*
*5. ———————— *Young.*

CASE F.

1. Ardea cinerea. *Heron.*
2. —— stellaris. *Bittern.*

CASE G.

*Ardea minuta. *Little Bittern.*

CASE H.

*Ibis falcinellus. *Ibis.*

CASE XII.

1 & 2. Recurvirostra Avocetta. *Avoset.*
3. Numenius arquata. *Curlew.*
4. ——— phæopus. *Whimbrel.*
5. Scolopax Rusticola. *Woodcock.*
6. ——— major. *Great Snipe. Female.*
7. ———————— *Male.*

8. Scolopax Gallinago. *Common Snipe. Male.*
9. ——— ——— ——— *Female.*
10. ——— Gallinula. *Jack Snipe.*
11. Limosa melanura. *Black-tailed Godwit. Spring plumage.*
12. ——— ——— ——— *Young First Autumn.*
13. ——— rufa. *Barred-tail Godwit. Spring plumage.*
14. ——— ——— ——— *Male. Summer plumage.*
15. ——— ——— ——— *Female. Winter plumage.*
16. Totanus Glottis. *Greenshank. Spring plumage.*
17. ——— ——— *Winter plumage.*
18. ——— fuscus. *Spotted Redshank. Young in Autumn.*
19. ——— ——— ——— *Adult in Winter.*
20. ——— Calidris. *Common Redshank. Summer plumage.*
21. ——— ——— ——— *Female. Winter plumage.*
22. ——— ——— ——— *Spring plumage.*
23. Tringa cinerea. *Knot. Summer plumage.*
24. ——— ——— *Spring plumage.*
25. ——— ——— *Autumnal plumage.*
26. ——— ——— *Winter plumage.*
27. ——— ——— *Young. First Autumn.*

Case XIII.

1. Tringa variabilis. *Purre. Young bird of the year.*
2. ——— ——— *Adult. Summer plumage.*
3. ——— ——— *Autumnal plumage.*
4. ——— ——— *Winter plumage.*
5. ——— maritima. *Purple Sandpiper.*
6. ——— pygmæa. *Pigmy Curlew. Winter plumage.*
7 & 8. ——— ——— *Summer*
9. ——— ——— *Spring plumage.*
10. ——— ——— *Young.*
11. Tringa minuta. *Little Sandpiper. Autumnal plumage.*
12. ——— ——— *Female. Winter plumage.*
13. ——— Temminckii. *Temminck's Sandpiper.*
14, 17, 18 & 20. Machetes pugnax. *Ruff. Summer plumage.*
15 & 16. ——— ——— *Winter plumage.*
19. ——— ——— *Reeve.*
21. Totanus glareola. *Wood Sandpiper. Female.*
22. ——— ——— *Male.*
23. ——— ochropus. *Green Sandpiper.*
*24. ——— macularia, *Spotted Sandpiper.*
25. ——— hypoleucos. *Common Sandpiper.*
26. ——— ——— *Female.*
27. Strepsilas Interpres. *Turnstone. Male.*
28. ——— ——— *Female.*
29. ——— ——— *Young Male.*
30. Vanellus cristatus. *Lapwing.*
31. ——— melanogaster. *Grey Plover. Spring plumage.*
32. ——— ——— *Male. Summer plumage.*
33. Charadrius pluvialis. *Golden Plover. Female. Winter plumage.*
34. ——— ——— *Male.*
35. Calidris arenaria, *Sanderling. Spring plumage.*
36. ——— ——— *Summer plumage.*
37. ——— ——— *Winter plumage.*
38. ——— ——— *Autumnal plumage.*

Case XIV.

1. Charadrius Morinellus. *Dotterell.*
*2. Himantopus melanopterus. *Long-legged Plover.*
3. Charadrius Hiaticula. *Ringed Plover. Female.*
4. ——— ——— ——— *Male.*
5 & 6. ——— cantianus. *Kentish Plover.*
7 & 8. Hæmatopus ostralegus. *Oyster-catcher.*
*9. Glareola torquata. *Pratincole.*
10 & 11. Rallus aquaticus. *Water Rail.*
12 & 13. Gallinula chloropus. *Moor-hen.*

14. Ortygometra Crex. *Land-Rail. Female.*
15. ——————————— *Male.*
16. Gallinula porzana. *Spotted Gallinule Female.*
17. ——————————— *Male.*
18. Gallinula Baillonii. *Baillon's Gallinule. Adult Male.*
19. ——————————— *Young Female.*
20. Phalaropus platyrhynchus. *Grey Phalarope. Winter plumage.*
21. Fulica atra. *Coot.*

Case XV.

1. Cygnus canadensis. *Canada Goose.*
2. Anser albifrons. *White-fronted Goose. Adult.*
3. ——————————— *Young.*
4. —— Segetum. *Bean Goose.*
5. —— egyptiacus. *Egyptian Goose.*
6. Bernicla erythropus. *Bernacle Goose.*

Case XVI.

1. Fuligula ferina. *Pochard. Male.*
2. ——————————— *Female.*
3. Rhynchaspis clypeata. *Shoveller. Male.*
4. ——————————— *Female.*
5. Querquedula Circia. *Gargany. Male.*
6. ——————————— *Female.*
•7. Somateria spectabilis. *King Duck. Male.*
•8. ——————————— *Female.*
9. Querquedula Crecca. *Teal. Male.*
10. ——————————— *Female.*
11. Fuligula cristata. *Tufted Duck. Male.*
12. ——————————— *Female.*
13. Harelda glacialis. *Long-tailed Duck. Male. Winter plumage.*
14. ——————————— *Female. Winter plumage.*
15. Dafila caudacuta. *Pintail. Male.*
16. ——————————— *Female.*

Case I.

Fuligula leucophthalmos. *Ferruginous Duck.*

Case XVII.

1. Cygnus ferus. *Wild Swan. Female.*
2. Tadorna Bellonii. *Shieldrake.* [*Young.*
3. ——————————— *Female and*
4. Mergus Merganser. *Goosander. Male.*
5. ——————————— *Female.*
6. ——— Serrator. *Red-breasted Goosander. Male.*
7. ——————————— *Female.*
8. Clangula chrysophthalmos. *Golden-eye. Male.*
9. ——————————— *Female.*
10. Mergus albellus. *Smew. Male.*

Case XVIII.

1. Podiceps cristatus. *Crested Grebe. Adult.*
2 & 3. ——————————— *Young.*
4. ——— rubricollis. *Red-necked Grebe. Young.*
5. ——— cornutus. *Sclavonian Grebe. Adult. Male.*
6. ——— obscurus. *Dusky Grebe.*
7 & 8. —— minor. *Little Grebe. Summer plumage.*
9 & 10. ——————————— *Winter plumage.*

Case K.

1. Uria Troile. *Foolish Guillemot.*
2. —— Grylle. *Black Guillemot.*

Case L.

Uria Alle. *Little Auk. Adult in Summer.*

Case M.

Uria Alle. *Little Auk. Young.*

Case N.

Alca Torda. *Razor Bill. Young.*

Case XIX.

1. Sula alba. *Gannet. Adult.*
2. ——————————— *Young.*
3. Carbo Cormoranus. *Cormorant. Young.*
4. —— Graculus. *Shag. Adult in Summer.*
5. —— Cormoranus. *Cormorant. Adult.*

Case XX.

1. Sterna cantiaca. *Sandwich Tern. Adult. Summer plumage.*
2. ——————————— *Winter plumage.*
3. ——————————— *Young. First Autumn.*
4 & 5. —— Hirundo. *Common Tern. Adult.*
6 & 7. ——————————— *Young.*
8. —— arctica. *Arctic Tern.*

9 & 10. Sterna nigra. *Black Tern.*
11 & 12. —————————— *Young.*
13 & 14. —— minuta. *Lesser Tern.*
15. —————————— *Young.*
16. Thalassidroma Leachii. *Leach's Petrel.*
17 & 18. ——— pelagica. *Stormy Petrel.*

Case XXI.

1. Larus marinus. *Great Black-backed Gull. Adult.*
2. —————————— *Young. Second year.*
3. —— glaucus. *Glaucous Gull. Young. First year.*
4.
5. —— fuscus. *Lesser Black-backed Gull. Adult.*
6. —————————— *Young. Second year.*
7. —— argentatus. *Herring Gull. Adult.*
8. —————————— *Young. Second year.*
9. —————————— *Nestling.*

Case XXII.

*1. Larus eburneus. *Ivory Gull.*
2. —— tridactylus. *Kittiwake. Adult in Winter.*
3. —————————— *Young.*
4. —— canus. *Common Gull. Young. First Winter.*
5. —— ridibundus. *Laughing Gull. Second Summer; and Nestling.*
6. —— canus. *Common Gull. Adult in Summer.*
7. —— ridibundus. *Laughing Gull. Spring of second year.*
8. —— tridactylus. *Kittiwake. Adult in Summer.*
9. —— ridibundus. *Laughing Gull. Winter plumage.*
*10. Lestris Catarractes. *Skua Gull.*
*11. ——— parasiticus. *Arctic Gull. Adult Male.*
12. —————————— *Young Male.*
*13. —————————— *Intermediate between Nos. 11 and 12.*

Desiderata

TO THE

COLLECTION OF BRITISH BIRDS

IN THE

MUSEUM OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

Circus cinerarius. *Ash-coloured Harrier.*
Sylvia suecica. *Blue-throated Warbler.*
Anthus Richardi. *Richards' Pipit.*
Plectrophanes lapponica. *Lapland Bunting.*
Alauda rubra. *Red Lark.*
Loxia Pityopsittacus. *Parrot Crossbill.*
Picus villosus. *Hairy Woodpecker.*
Columba Livia. *Rock-Dove.*
Phasianus torquatus. *Ring-necked Pheasant.*
Ardea purpurea. *Purple-Heron.*
—— Egretta. *Great white Heron.*
—— æquinoctialis. *Little white Heron.*
—— Ralloides. *Squacco Heron.*
—— lentiginosa. *Freckled Heron.*
Platalea Leucorodia. *Common Spoonbill.*
Ciconia alba. *White Stork.*
—— nigra. *Black Stork.*
Scolopax Sabini. *Sabine's Snipe.*
—— grisea. *Brown Snipe.*
Tringa rufescens. *Buff-breasted Sandpiper.*
Lobipes hyperboreus *Red Phalarope.*
Gallinula pusilla. *Little Gallinule.*
Cursorius isabellinus. *Cream-Coloured Courser.*
Plectropterus gambensis. *Spur-winged Goose.*
Anser ferus. *Wild Goose.*
Bernicla ruficollis. *Red-breasted Goose.*
Tadorna rutila. *Ruddy Goose.*
Anas strepera. *Gadwall. Female.*
Querquedula glocitans. *Bimaculated Duck.*
Clangula histrionica. *Harlequin Duck.*
Fuligula rufina. *Red-crested Pochard.*
—— Marila. *Scaup Duck.*
Somateria molissima. *Eider Duck.*
Oidemia nigra. *Scoter Duck.*
—— fusca. *Velvet Duck.*
—— leucocephala. *White-headed Duck.*
—— perspicillata. *Black Duck.*
Podiceps auritus. *Eared Grebe.*
—— rubricollis. *Red-necked Grebe. Adult.*
Colymbus glacialis. *Northern Diver.*
—— arcticus. *Black-throated Diver.*
—— septentrionalis. *Red-throated Diver.*
Alca impennis. *Great Auk.*
Carbo cristatus. *Crested Shag.*
Sterna Dougallii. *Roseate Tern.*
—— anglica. *Gull-billed Tern.*
Larus glaucus. *Glaucous Gull. Adult.*
—— islandicus. *Iceland Gull.*
—— capistratus, (*Temm.*)
—— atricilla, (*Temm.*)
—— minutus. *Little Gull.*
Lestris pomarinus. *Pomarine Gull.*
Procellaria glacialis. *Fulmar.*
Puffinus Anglorum. *Shearwater.*

N.B.—Mr. Leadbeater, No. 19, Brewer Street, Golden Square, is employed by the Society for stuffing their Birds, and will prepare any of the above for them, if sent to him. At the same time, notice of any Bird presented should be addressed to the Secretary of the Society at Cambridge.

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

VOL. III. PART II.

XIII. *On a Correction requisite to be applied to the Length of a Pendulum consisting of a Ball suspended by a fine Wire.*

By GEORGE BIDDELL AIRY, M.A.

MEMBER OF THE ASTRONOMICAL SOCIETY, FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, AND PLUMIAN PROFESSOR OF ASTRONOMY AND EXPERIMENTAL PHILOSOPHY, IN THE UNIVERSITY OF CAMBRIDGE.

[Read *Nov.* 16, 1829.]

In the deduction of the length of the simple pendulum vibrating seconds from the time of vibration of a compound pendulum, consisting of a metallic ball supported by a wire, it has always been supposed that the diameter of the ball which in its position of rest was vertical, continues during the whole vibration to be in the same straight line as the wire. This at least was tacitly assumed in the experiments by Borda, Cassini, Arago, and Biot, (the corrections in the *Base du Systeme Metrique*, Tome III. p. 358, and Biot, *Astronomie Physique*, Tome III. Additions p. 173, are calculated on that supposition,) and in the only account which I have seen of the experiments lately made by Mr. Bessel, the same thing is tacitly supposed to be correct. Yet it is perfectly certain that, during the vibration, the wire and the diameter which was vertical will make an angle, except in one position which will in practice be the same as the position of rest. I propose in this paper to investigate the motion of such a pendulum,

and to examine particularly whether, with the dimensions which have been used or may probably again be used, the neglect of the relative rotation of the ball and wire will introduce any sensible error in the concluded length of the seconds' pendulum.

The force which causes the ball to move horizontally is the resolved part of the tension of the wire. Suppose for the sake of simplicity the arc of vibration to be so small that the tension of the wire may be considered constant. Then the motion of translation of the ball will be occassioned by a force proportional to the distance of the point of attachment of the wire from the vertical: the motion of rotation will depend upon the whole tension, and upon the angle made by the wire produced with the diameter which was vertical (as the other force which acts on the ball, namely, its weight, may be considered as acting at its center of gravity.)

Let a be the length of the wire, from the point of suspension to the point where it is attached to an inflexible part of the ball or bob: r the distance of the ball's center of gravity from the same point: k the distance of the center of gyration; θ the angle made by the wire with the vertical: ϕ the angle made with the vertical by that line in the ball which in the position of rest was vertical; M the mass of the ball: which will also represent the tension of the wire. Then the horizontal force acting on the ball is $M.\sin\theta$: and the equation for the motion of translation gives

$$\frac{d^2}{dt^2}(a\sin\theta + r\sin\phi).M = -gM\sin\theta.$$

The momentum of the force of tension about its center of gravity is $Mr\sin(\theta-\phi)$, and the equation of rotation gives

$$k^2 M\frac{d^2\phi}{dt^2} = gMr\sin(\theta-\phi).$$

Putting the arcs for the sines we have these simultaneous equations

$$a\frac{d^2\theta}{dt^2} + r\frac{d^2\phi}{dt^2} = -g\theta,$$

$$\frac{d^2\phi}{dt^2} \qquad = \frac{gr}{k^2}(\theta - \phi).$$

If we multiply the second equation by m and add it to the first, we have

$$a\frac{d^2\theta}{dt^2} + (r+m)\frac{d^2\phi}{dt^2} = -g\left(1 - \frac{mr}{k^2}\right)\theta - g\frac{mr}{k^2}\phi.$$

$$\text{Let } \frac{1 - \frac{mr}{k^2}}{a} = p, \quad \text{or } m = \frac{k^2}{r} - \frac{ak^2}{r}p:$$

$$\text{then } a\frac{d^2\theta}{dt^2} + \left(r + \frac{k^2}{r} - \frac{ak^2}{r}p\right)\frac{d^2\phi}{dt^2} = -gap\theta - g(1-ap)\phi.$$

This will be integrable, if

$$\frac{r}{a} + \frac{k^2}{ar} - \frac{k^2}{r}p = \frac{1-ap}{ap}:$$

$$\text{or } p^2 - \left(\frac{1}{a} + \frac{r^2}{ak^2} + \frac{r}{k^2}\right)p + \frac{r}{ak^2} = 0.$$

Let p' and p'' be the two roots of this equation, which are both possible and positive, m' and m'' the corresponding values of m. Then the differential equation, upon substituting these values, takes the forms

$$\frac{d^2}{dt^2}\left(\theta + \frac{r+m'}{a}\phi\right) = -gp'\left(\theta + \frac{r+m'}{a}\phi\right)$$

$$\frac{d^2}{dt^2}\left(\theta + \frac{r+m''}{a}\phi\right) = -gp''\left(\theta + \frac{r+m''}{a}\phi\right).$$

Their solution gives

$$\theta + \frac{r+m'}{a}\phi = A'\cos\left(t\sqrt{gp'} + B'\right)$$

$$\theta + \frac{r+m''}{a}\phi = A''\cos\left(t\sqrt{gp''} + B''\right),$$

and by solving the two simple equations we find that θ and ϕ are expressed by the forms

$$C' \cos(t\sqrt{gp'} + B') + C'' \cos(t\sqrt{gp''} + B''),$$

$$\text{and} \quad c' \cos(t\sqrt{gp'} + B') + c'' \cos(t\sqrt{gp''} + B''),$$

where two of the constants C', C'', c', c'' are arbitrary. The motion of the wire, and the rotation of the ball in alternate directions, may therefore be represented by the superposition of two vibrations, each of which follows the law of the cycloidal pendulum.

Now it is easily seen that one of these vibrations is of that kind which will take place if, without giving any motion to the center of gravity of the ball, it be turned round a horizontal axis and then be set at liberty: and this is performed in a short period. The other is of the kind which is commonly considered, and except the disturbance of the ball be very great, is the only one that catches the eye in vibration. This then is the only one which is used in the observations of the pendulum: it is therefore the only one which concerns us here. It may be distinguished from the other by observing that it is performed in a longer period, and therefore that value of p must be taken which requires the greatest value of t to make the term $A\cos(t\sqrt{p} + B)$ go through all its periodical values: that is, we must take the smaller value of p. Let this be p'. Then the apparent time of a double vibration will be $\frac{2\pi}{\sqrt{gp'}}$: or the length of the simple pendulum which would vibrate in the same time will be $\frac{1}{p'} = \frac{ak^2}{r}p''$. Call this l.

$$\text{Now } p'' = \tfrac{1}{2}\left(\frac{1}{a} + \frac{r^2}{ak^2} + \frac{r}{k^2}\right) + \tfrac{1}{2}\sqrt{\left\{\left(\frac{1}{a} + \frac{r^2}{ak^2} + \frac{r}{k^2}\right)^2 - \frac{4r}{ak^2}\right\}},$$

$$\therefore\ l = \tfrac{1}{2}\left(a + r + \frac{k^2}{r}\right) + \tfrac{1}{2}\sqrt{\left\{\left(a + r + \frac{k^2}{r}\right)^2 - \frac{4ak^2}{r}\right\}}.$$

Let l' be the length as commonly estimated: that is, let

$$l' = a + r + \frac{k^2}{a+r}.$$

Then

$$l - l' = \tfrac{1}{2}\sqrt{\left\{\left(a + r + \frac{k^2}{r}\right)^2 - \frac{4ak^2}{r}\right\}} - \tfrac{1}{2}\left(a + r - \frac{k^2(a-r)}{r(a+r)}\right),$$

$$= \frac{\dfrac{2ak^4}{r(a+r)^2}}{\sqrt{\left\{\left(a + r + \dfrac{k^2}{r}\right)^2 - \dfrac{4ak^2}{r}\right\}} + \left(a + r - \dfrac{k^2(a-r)}{r(a+r)}\right)},$$

$$= \frac{ak^4}{r(a+r)^3} \text{ nearly} = \frac{k^4}{l^2 r} \text{ with sufficient accuracy for practice.}$$

The length of the simple pendulum which would vibrate in a given observed time, is therefore greater than it is commonly inferred from observations with this particular apparatus; and consequently the length of the seconds' pendulum is greater than it is estimated, in the same ratio. Let L be the length of the seconds' pendulum: then we ought to add to the estimated length $\frac{Lk^4}{l^3 r}$.

In this investigation it will be observed that the consideration of spherical form has not occurred: and that r is the distance from the center of gravity to the point where the wire is firmly attached to an inflexible part of the ball. If the ball be a sphere of radius R, $k^2 = \frac{2}{5} R^2$, and the quantity to be added is

$$\frac{4}{25} \cdot \frac{L \cdot R^4}{l^3 r}.$$

In Borda's experiments $l = 12$ feet $= 1728$ lines; $R = 8$ lines: $L = 440$ lines: r was somewhat greater than R; perhaps $= 12$ lines. Consequently the quantity to be added is about $\frac{1}{200,000}$ line: a quantity quite insensible.

In Biot's experiments, the length of the wire was about $\frac{1}{4}$ that in Borda's, and therefore supposing the balls of the same

diameter, the allowance to be made must be 64 times as great, or $\frac{1}{3,000}$ line: a quantity perhaps sensible, or which would at least alter the figures in the last decimal places of Biot's estimations.

Had the diameter of the balls been twice as great, in the last experiment, the error would have been of a magnitude greater than could occur in any comparison of standards.

It appears then that by a happy chance the error of estimation arising from the neglect of this consideration has not yet been sensible, though it has closely approached the limit of perceptible error. As a guide in future experiments, I shall give a short statement of errors corresponding to given magnitudes of the spheres, supposing $\frac{R}{r} = \frac{2}{3}$.

The length of the seconds' pendulum being estimated in the usual way, the quantity to be added in inches is $4,27 \times \left(\frac{R}{l}\right)^3$.

		inch.
If $\frac{R}{l} = ,01$	this is	0,000004,
$\frac{R}{l} = ,02$		0,000034,
$\frac{R}{l} = ,03$		0,000115,
$\frac{R}{l} = ,04$		0,000273,
$\frac{R}{l} = ,05$		0,000534,
$\frac{R}{l} = ,1$		0,00427.

G. B. AIRY.

OBSERVATORY, *July* 2, 1829.

XIV. *On an Ancient Observation of a Winter Solstice.*

BY R. W. ROTHMAN, ESQ. M.A.

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

[Read *Nov.* 30, 1829.]

STRABO in the second book of his Geography has recorded an observation of the Winter Solstice made at Alexandria, which, as far as I know, has never hitherto been noticed. This may be attributed to an extraordinary mistake, by which Strabo has given this as an observation of the equinox. The commentators on this author have perceived that there was a mistake, without seeing the nature of it: and (vide *Commentary of the Variorum Edition, Leipsic,* 1818. Vol. VII. p. 573.) M. Gosselin has proposed to make an alteration in the numbers. But I think I shall be able to shew to the satisfaction of the Society that the error lies in the observation being given as belonging to the equinox, while in reality it belongs to the winter solstice. I shall just remark in passing that M. Delambre in his *Astron. Ancienne,* Vol. I. p. 257, has quoted the passage in question from Strabo without any remark.

The words of Strabo are as follows, *ἐν δὲ τῇ Ἀλεξανδρείᾳ ὁ γνώμων λόγον ἔχει πρὸς τὴν ἰσημερινὴν σκιάν, ὃν ἔχει τὰ πέντε πρὸς ἑπτά,* (Vide Strabo, Lib. II. *Ed. Varior. Leipsic,* 1796.

Vol. I. p. 355.) or "at Alexandria, the gnomon has the same ratio to the equinoxial shadow that 5 has to 7."

Now

log. 7	=	0.8450980
log. 5	=	0.6989700
log. tan. z	=	0.1461280

(where z = zenith distance observed,)

∴ z	=	54°.27'.45"
Refraction	=	0 . 1.19
Parallax	=	− 0 . 0. 6
		54.28.58
½ Sun's diameter	=	0.15.58
Corrected zenith distance	=	58.44.56

Now by modern observations the latitude of Alexandria is 31°.13'.5". Vide positions given in *Tables du Bureau des Longitudes*, Tab. I. Adding then to our corrected zenith distance the co-latitude of the place, we shall soon see, that the zenith distance in question evidently corresponds to the winter solstice: for

Co-latitude of Alexandria	=	58°.46'.55"
Zenith distance of Sun	=	54.44.56
Sum	=	113.31.51
		90. 0. 0
		23.31.51

If we subtract from the sum 90° we find the Sun to the southward of the equator 23°.31'.51": which proves clearly the observation to have been made at the winter solstice.

It may perhaps be interesting to examine how far this observation agrees with the formula founded on the Theory of Universal Gravitation for the diminution of the obliquity of the ecliptic. If we take the constants as given by Poisson, *Con. des Temps.* 1830, Additions, p. 29, we shall have the obliquity of the ecliptic at any number of years t from the year 1750, expressed by this formula

$$23^{\circ}.\,28'.\,18'' - t\,.\,0''.45692 - t^2.0''.000002242.$$

Taking for t, − 1750, which will bring us to the age of Strabo, this gives us for the calculated obliquity

	=	23°. 41′. 30″
Observed obliquity	=	23 . 31 . 51
Difference	=	0 . 9 . 39

Though this difference of nearly 10′ may not perhaps appear very great, considering the instrument employed: yet it seems to exceed the probable error of these observations. The observation of Pytheas in Strabo, gives the latitude of Marseilles within four minutes: that of Eratosthenes given by Cleomedes, gives the latitude of Alexandria, within five minutes. In this case it is not improbable that finding the ratio of the gnomon to the solsticial shadow very nearly equal to 5 : 7, the Greeks have voluntarily neglected the trifling difference, in order to express this ratio in the simplest terms.

R. W. ROTHMAN.

TRINITY COLLEGE,
Nov. 27, 1829.

XV. *On the Crystals of Boracic Acid, &c.*

By W. H. MILLER, Esq. M.A.

FELLOW OF ST. JOHN'S COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *Nov.* 30, 1829.]

As the following crystals do not appear to have been determined in any work on crystallography, I beg to offer the Society the results of my own observations upon them. All the measurements were taken with the reflective goniometer of Wollaston.

I. *Boracic Acid.*

Primary form, a doubly oblique prism.

$$
\begin{aligned}
PM &= 80^{\circ}.30' \\
PT &= 84\ .53 \\
MT &= 118\ .30 \\
Pk &= 75\ .30 \\
Mk &= 120\ .45 \\
T'k &= 120\ .45 \qquad \text{(Fig. 1.)} \\
Pc &= 129 \\
Pe &= 132 \\
Pf &= 139 \\
Ph &= 137 \\
Px &= 150 \\
Py &= 156
\end{aligned}
$$

the planes *c, e, f, h, x, y*, seldom occur, and are too imperfect to be measured accurately by the reflective goniometer, the angles *PC*, &c. are therefore given to the nearest degree only.

Cleavage, parallel to *P*; very perfect.

Twin crystals are common, having the axis of revolution parallel to the intersection of *M* and *T*, and the face of composition parallel to *k*. $PP_{,} = 150°.58'$. (Fig. 2.)

The axes of double refraction intersect the plane *P*, and make with each other an angle of about 8° or 9°.

The crystals were deposited from a solution of the acid in water.

II. *Borate of Ammonia.*

Primary form, a square prism.

$$aa_{,} = 115°.13'$$
$$aa_{,,} = 98\ .30 \qquad \text{(Fig. 3.)}$$
$$MM_{,} = 90$$

Twin crystals occur having the axis of revolution parallel to *M*, and perpendicular to the intersection of *a* and $a_{,}$. (Fig. 4.) represents an assemblage of twin crystals.

The crystals were obtained from a solution of the salt in water.

III. *Indigo.*

Primary form, a right rhombic prism.

$$MM_{,} = 103°.30'$$
$$Mh = 128\ .15$$
$$Mg = 149\ .12 \qquad \text{(Fig. 5.)}$$
$$gg_{,} = 165\ .\ 6$$
$$cc_{,} = 108$$
$$hc = 126$$

The crystals were obtained by sublimation.

IV. *Bicarbonate of Ammonia.*

Primary form, a right rhombic prism.

$MM_{,}$	$= 111^{\circ}.48'$	
PM	$= 90$	
Ph	$= 90$	
$cc_{,}$	$= 135\ .40$	(Fig. 6.)
$aa_{,}$	$= 117\ .40$	
Mh	$= 124\ .\ 6$	
Pc	$= 157\ .50$	
Pa	$= 148\ .50$	

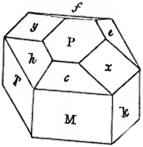
Cleavage, parallel to M and $M_{,}$; very perfect.

The crystals were deposited from a saturated solution in a close vessel.

W. H. MILLER.

ST. JOHN'S COLLEGE,
Nov. 30, 1829.

1



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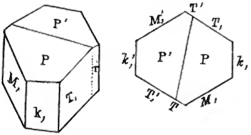

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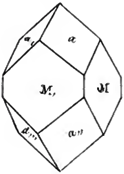

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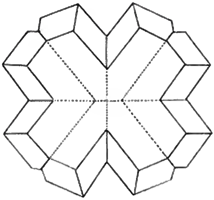

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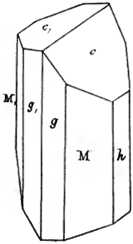

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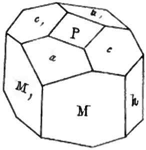

XVI. *On certain Conditions under which a Perpetual Motion is possible.*

By GEORGE BIDDELL AIRY, M.A.

MEMBER OF THE ASTRONOMICAL SOCIETY, FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, AND PLUMIAN PROFESSOR OF ASTRONOMY AND EXPERIMENTAL PHILOSOPHY, IN THE UNIVERSITY OF CAMBRIDGE.

[Read *Dec.* 14, 1829.]

It is well known that *perpetual motion* is not possible with any laws of force with which we are acquainted. The impossibility depends on the integrability *per se* of the expression $Xdx + Ydy + Zdz$: and as in all the forces of which we have an accurate knowledge this expression is a complete differential, it follows that perpetual motion is incompatible with those forces.

But it is here supposed that, the law of the force being given, the magnitude of the force acting at any instant depends on the position, at that instant, of the body on which it acts. If however the magnitude of the force should depend not on the position of the body at the instant of the force's action, but on its position at some time preceding that action, the theorem that we have stated would no longer be true. It might happen that, every time that the body returned to the same position, its velocity would be less than at the preceding time: in this case the body's motion would ultimately be destroyed. On the contrary it might happen that the body's velocity in any position

would be more rapid every time than at the time previous. In this case the velocity would go on perpetually increasing: or the velocity might be made uniform if the machine were retarded by some constantly acting resistance: or, in other words, the machine might move with uniform velocity, and might at the same time *do work:* which is commonly understood to be the meaning of the term *perpetual motion*. If the machine had no work to do, the increasing friction, &c. would operate as an increasing work, and the velocity would be accelerated till the acceleration caused by the forces was equal to the retardation caused by the friction: after which it would remain unaltered.

For this idea I am indebted to the admirable account of the organs of voice given by Mr. Willis. The phenomenon to be explained was this. When two plates are inclined at an angle greater than a certain angle, it is found that the effect of a current of air passing between them is to give a tendency to open wider. When they are inclined at any angle smaller than that certain angle, the effect of the current is to make them collapse. If then the plates be supposed to vibrate through the position corresponding to that angle, the tendency of the forces at all times is to bring them to that position. Each plate is in the state of a vibrating pendulum: and whatever be the law of force which acts on it it is certain that if the force be the same when the plate is in the same position, this force will have no tendency to increase the velocity. The retardation arising from friction, &c. will therefore soon destroy the motion. But it is found, in fact, that the motion is not destroyed. What then is the accelerating force which keeps up the motion? Mr. Willis explains this by supposing that *time* is necessary for the air to assume the state and exert the force corresponding to any position of the plate: which is nearly the same as saying that the force depends on the position of the plate at some previous time. In this

Paper, which is intended to investigate the mathematical consequences of an assumed law, I shall not discuss the identity of these suppositions: I shall only remark that the general explanation appears to be correct, and that it clears up several points which always appeared to be in great obscurity.

Let us now consider the case of a vibrating body acted on by two forces, of which one is proportional to its actual distance from the point of rest, and the other proportional to its distance at some previous time. Putting $\phi(t)$ for the body's distance, the equation is

$$\frac{d^2 . \phi(t)}{dt^2} = -e . \phi(t) - g . \phi(t-c).$$

This equation I am unable to solve rigorously: but on the supposition that g is small, an approximate solution may be obtained from the formulæ in the Memoir on the Disturbances of Pendulums, &c. (*Cam. Trans.* Vol. III. p. 109.) Neglecting at first the small term, we have

$$\frac{d^2 \phi(t)}{dt^2} = -e . \phi(t),$$

whence

$$\phi(t) = a . \sin(t\sqrt{e} + b).$$

Consequently

$$\phi(t-c) = a . \sin(t\sqrt{e} + b - c\sqrt{e}),$$

and therefore f in the formulæ alluded to is

$$= ag . \sin(t\sqrt{e} + b - c\sqrt{e}).$$

The increase therefore of the arc of semi-vibration is

$$-\frac{1}{\sqrt{e}} \int_t ag . \sin(t\sqrt{e} + b - c\sqrt{e}) \cos(t\sqrt{e} + b)$$

$$= -\frac{ag}{2\sqrt{e}} \int_t \{\sin(2t\sqrt{e} + 2b - c\sqrt{e}) - \sin c\sqrt{e}\}.$$

To find the increase from one vibration to another we must take the integral between two values of t differing by $\frac{2\pi}{\sqrt{e}}$: and thus we obtain for the increase $\frac{\pi \, . \, ag \sin c\sqrt{e}}{e}$.

I shall not occupy the time of the Society by a discussion of the different values of the increase corresponding to different values of c: I shall only remark that if $c\sqrt{e}$ be less than π, the arc of vibration increases continually. Nor shall I consider the cases in which c is supposed to be a function of the position or velocity of the vibrating body (which possibly might better represent the circumstances that originally suggested this investigation.) My object is gained if I have called the attention of the Society to a law hitherto (I believe) unnoticed, but not unfruitful in practical applications.

G. B. AIRY.

OBSERVATORY, *Dec.* 13, 1829.

XVII. *Some Observations on the Habits and Character of the Natter-Jack of Pennant, with a List of Reptiles found in Cambridgeshire.*

By the Rev. LEONARD JENYNS, M.A. F.L.S.

AND FELLOW OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *Feb.* 22, 1830.]

It is observed by Dr. Fleming, in his *British Animals*, that "the history of the Natter-Jack, like that of many of our native reptiles, is involved in obscurity." Under the sanction of such a remark from one of our first Naturalists, I am induced to offer a few observations which I have had an opportunity of making upon the habits of this species, and to record it as a native of Cambridgeshire.

This animal does not appear to have been often noticed in this country. Its first discoverer, I believe, was the late Sir Joseph Banks, who found it near Revesby Abby in Lincolnshire, and sent an account of it to Pennant, stating that it was known in the above district by the name of *Natter-Jack*. This account was afterwards published by Pennant in his *British Zoology*, (Vol. III. p. 19.) who also says that it occurs on Putney Common.—I am not aware that it has been mentioned by any other British writer, otherwise than with reference to Pennant's original description.

The first instance in which this reptile occurred to my notice was in August 1824, when it was discovered by Professor Henslow and myself, in considerable abundance, in the bogs upon Gamlingay Heath. Soon after our visit to that spot, a single individual was found by myself at Bottisham, and a second has since been met with by Professor Henslow in the Botanic Garden. I am now inclined to the opinion, that it is not so very local a species as was formerly supposed, but that from its general resemblance to the common Toad, it has been often overlooked.

The individuals found at Gamlingay, and which are the subject of the following remarks, I brought away with me from that place alive, and succeeded in keeping in that state during a period of nearly two months. For the first fortnight after their confinement, these animals refused every kind of food that was offered to them; nor could I perceive that they ate any thing which happened to fall in their way, though they retained both their plumpness and activity. At the end of that period, however, they became more reconciled to their situation, and readily devoured flies and other insects that were placed before them, although it was absolutely necessary that these should be given them alive: indeed in no instance could they be induced to touch their prey, till it began to move, and to shew signs of preparing to escape. Their manner of seizing their food was very curious. As soon as an insect was thrown down into the cage in which they were kept, the first individual that saw it immediately pricked up his head, turned quickly round, and ran towards it till it got within a certain distance, when it would again stop,—crouch down upon its belly with its hind legs stretched out, and gaze at it with all the silent eagerness of a staunch pointer. In this position it would always remain till its prey began to move, when, just as the victim was about to make its escape, it would suddenly dart out its tongue, and lick it up with a rapidity too

quick for the eye to follow. Sometimes, however, especially if the insect were nimble, it would follow it about the cage for a considerable length of time before it would attempt to secure it, stopping every now and then to gaze at it, apparently with much delight, for many seconds together. Nor, in its endeavour to seize its food, was it always able to measure its distance with correctness, often falling short of its aim, and making two or three fruitless attempts, before it was finally secured. When, however, this was once accomplished, the booty was swallowed instantly, excepting when above a certain size, in which case, the Natter-Jack would occasionally remain for ten minutes afterwards with one half of the insect in its throat, and the other hanging out of its mouth.

With respect to the nature of the food devoured by these animals, I may observe, that they seemed to relish most, the smaller species of Diptera and Hymenoptera, though they would occasionally take woodlice and even centipedes. They also ate large quantities of a small red maggot which is generally abundant in decayed Boleti, and any of the lesser Coleoptera which might happen to stray into their cage. One of them, in a single instance, attacked an ant, but the morsel did not appear to be much relished, for it was no sooner conveyed to its mouth, than rejected with great haste and trepidation, probably in consequence of the strong acid which is secreted by these insects. They did not, however, appear to suffer from the stings of the smaller bees and ichneumons, which were repeatedly swallowed with impunity.

The Natter-Jack is a much more lively animal than the common Toad, and when in search of food, or following its prey, shews great alertness. When full fed, or from other causes inactive, the above individuals would conceal themselves in a sod of turf, which was always kept in their cage. They also

occasionally delighted much in a pan of water, in which they would float motionless for half an hour together, having all their legs stretched out, and no part of their body except their head above the surface. But the great distinguishing habit of this species is its mode of progression. Unlike the Frog, in this respect, which advances by regular leaps; and the Toad, whose pace is seldom exerted beyond that of a slow crawl; the Natter-Jack has a kind of shuffling run, which is seen to most advantage when it is following its prey, and by which means, it is enabled, when in full health and activity, to get over its ground with considerable quickness.

The general outward appearance of this animal is, as I before observed, similar to that of the common Toad; nevertheless, the following description will serve to discriminate it from that species.

Upper part of the body yellowish-brown, clouded here and there with shades of a darker colour, and covered with porous warts and pimples of various sizes, which are generally black, enclosing a red spot. A bright golden yellow line runs down the back, extending from the top of the head to the anus, and is very characteristic of the species. Over each shoulder, behind the eyes, is a slight dash of brick red. The under parts are thickly covered with warts of a whitish hue, which are large and more scattered towards the posterior, smaller and more crowded towards the anterior extremity. Besides these, the whole of the abdomen, the sides, and in some instances the breast and throat, are thinly spotted with black. Chin white. Eyes somewhat elevated, and projecting. Tongue connected with the lower jaw as far as the lip, from whence it extends into a kind of spatula which is folded back upon itself, when not in use. Feet spotted with black; the spots, in some instances, uniting to form transverse bands. The anterior pair with four divided

toes, of nearly equal length. The posterior, with five perfectly-formed, a little webbed, and the rudiment of a sixth*: of these the fifth is more than double the length of any of the others. Extremities of all the toes black.

The general colour of this animal varied in different individuals;—in some approaching to yellow,—in others almost to black. In such as were sickly, the black had a lurid dingy appearance;—the colours lost their brightness, and the yellow dorsal line became nearly obsolete. All the specimens which have as yet fallen under my notice have been small and considerably under size. Pennant states the following to be the dimensions of this reptile. "Length of the body two inches and a quarter; the breadth, one and a quarter: the length of the fore legs one inch and one-sixth; of the hind legs, two inches."

I shall conclude this paper with a brief enumeration of all the species belonging to this class of animals which I have hitherto discovered in Cambridgeshire.

ORDER I. Sauria.

Genus I. LACERTA. *Linn.*

Sp. 1. L. agilis, *Flem. Brit. Anim.* p. 150.

Common Lizard.—A very variable species, of which hardly two individuals are to be found, marked exactly in the same way. It is common every where in dry sunny situations, throughout the spring and summer months, and is in general first seen about

* In the common Toad, the sixth toe on the posterior pair of feet is much more developed than in the Natter-Jack. The toes are also more webbed.

the beginning of April;—though, in one instance, I noticed it as early as the twelfth of March. The young broods appear in July.

The tail of this animal as is well known, is extremely brittle, and a very slight blow or pressure is sufficient to cause it to separate immediately from the body. No blood issues from the wound, but the severed part will continue to move backwards and forwards, and to shew signs of life for a considerable time afterwards. It is, however, easily reproduced; and until this operation is effected, I am inclined to think that the individual retires usually to some place of concealment; having found it, under such circumstances, in a languid quiescent state beneath the bark of felled timber.

ORDER II. Ophidia.

Genus II. ANGUIS. *Linn.*

Sp. 2. A. fragilis, ***Flem. Brit. Anim.*** p. 155.

Common Blind-Worm.—This does not appear to be frequent in Cambridgeshire. I have only observed it in one or two instances in the neighbourhood of Bottisham.

Genus III. NATRIX. *Flem.*

Sp. 3. N. torquata, ***Flem. Brit. Anim.*** p. 156.

Ringed Snake.—This species, which is our common snake, appears to delight much in the water, and is particularly abundant in the fens, where it sometimes attains a large size. It is generally first seen about the beginning of April. On the twenty-second of that month, I have found the sexes in copulation; during which act, they are extended side by side in a straight line.

Genus IV. VIPERA. *Daud.*

Sp. 4. V. communis, *Flem. Brit. Anim.* p. 156.

Common Viper.—Very rarely met with in Cambridgeshire. It has never occurred to my certain knowledge at Bottisham; but I am informed by Dr. Haviland, that some years back a man was brought to Addenbrooke's Hospital, who had been bitten by a venomous snake at the back of Queen's College, which was supposed to have been of this species. It has also been met with in two other instances in the neighbourhood of Cambridge.

ORDER III. Batrachia.

Genus V. TRITON. *Lauren.*

Sp. 5. T. palustris, *Flem. Brit. Anim.* p. 157.

Warty Eft.—This species, which is very common in all our ditches in the spring months, may be often observed at other periods of the year, under stones and rubbish in damp places, in a state of quiescence. No doubt, in some instances, this is in consequence of the drying up of the waters in its accustomed haunts from the heat of the summer; but in others, it appears to be the result of choice. It is also occasionally found in cellars.

Sp. 6. T. aquaticus, *Flem. Brit. Anim.* p. 158.

Water Eft.—Smaller than the last species, from which it is easily distinguished by its comparatively smooth skin. Common in stagnant waters.

Sp. 7. T. vulgaris, *Flem. Brit. Anim.* p. 158.

Common or Land Eft.—This species is met with under stones, and in cellars; but, as far as my observation goes, is never found

in the water. I agree with the Rev. Revelt Sheppard, (*Linn. Trans.* Vol. VII. p. 55.) in believing that it undergoes no change, and that it is perfectly distinct from either of the foregoing species.

Genus VI. RANA. *Linn.*

Sp. 8. R. temporaria, *Flem. Brit. Anim.* p. 158.

Common Frog.—This well-known reptile spawns about the middle of March, and the young tadpoles are hatched a month or five weeks afterwards, according to the warmth of the season. By the eighteenth of June, I have observed these to be nearly full-sized, and beginning to acquire their fore-feet: and towards the end of that month or the beginning of the next, (varying in different years,) the young frogs may be seen in great numbers, forsaking the water in which they were bred, and coming on land.

Genus VII. BUFO. *Cuv.*

Sp. 9. B. vulgaris, *Flem. Brit. Anim.* p. 159.

Common Toad.—Where this and the preceding species spend the winter does not appear to have been satisfactorily ascertained. It is, however, a curious circumstance, that from the end of February to the beginning or middle of April, they are to be found in countless numbers at the bottoms of all our ditches, ponds, and other stagnant waters. During this interval, which is the period of the breeding season, they keep up a perpetual croaking, and the act of copulation, which lasts several days, is performed. From many years' observations, I find that the Toad is invariably a few days later in spawning than the Frog. In some seasons, this difference has amounted to more than a fortnight.

Sp. 10. B. Rubeta, *Flem. Brit. Anim.* p. 159.

NATTER-JACK.—Found, as stated in the former part of this paper, on Gamlingay Heath, at Bottisham, and also in the Botanic Garden at Cambridge.

From the small size of the individuals found at Gamlingay, of which some appeared to have not very long left the water and passed into a perfect state, together with the lateness of the season when they were observed,—I conclude that this species does not spawn so early in the year as either the common Toad or Frog.

L. JENYNS.

SWAFFHAM BULBECK,
Feb. 20, 1830.

XVIII. *On the General Equations of the Motion of Fluids, both Incompressible and Compressible; and on the Pressure of Fluids in Motion.*

By J. CHALLIS, M.A.

FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *Feb.* 22, 1830.]

I. *Incompressible Fluids.*

1. The general equations relating to the motion of incompressible fluids, are,

$$p = V - \frac{d\phi}{dt} - \frac{1}{2}\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) \qquad (1),$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0, \qquad (2),$$

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz} \qquad (3).$$

(Poisson, *Traitè de Mecanique*, Tom. II. p. 486.)

p is the pressure at any point, the co-ordinates of which are x, y, z; u, v, w, are the velocities in the directions of x, y, z, respectively; dV is put for $Xdx + Ydy + Zdz$, in which X, Y, Z, are the components in the directions of x, y, z, respectively, of

the accelerative forces impressed at the point; and this substitution is legitimate, because, as the forces of nature are directed to fixed or moveable centres, $Xdx + Ydy + Zdz$ will be generally a complete differential with respect to x, y, and z, of a function of x, y, z, and t: ϕ is a function of x, y, z, and t, which was introduced in the course of the investigation of the preceding equations, by substituting $d\phi$ for $udx+vdy+wdz$; for it appeared that these equations do not admit of a simple form, unless $udx+vdy+wdz$ be a complete differential of a function of x, y, and z, which may also contain t, but is not differentiated with respect to this variable. Consequently the equations (1), (2), (3) cannot be made use of, except in cases in which we are assured that this condition is satisfied.

It has occurred to me that the analytical fact, that the equations of the motion admit of simplification when $udx+vdy+wdz$ is a complete differential of a function of x, y, z, has reference to the manner of action of the parts of the fluid on each other. If this be such that the motion in every elementary portion is directed to a fixed or moveable centre, $udx+vdy+wdz$ will be a complete differential of a function of x, y, z, for the same reason that $Xdx+Ydy+Zdz$ is a complete differential of the same variables. And it is possible to obtain an integral of (2), which will accord with this character of the motion. For suppose ϕ to be a function of r and t, r^2 being equal to $x^2+y^2+z^2$. Then by substituting in (2) we obtain $\frac{d^2.r\phi}{rdr^2}=0$, an equation which is not contradictory to the supposition. In consequence of the supposed nature of the function ϕ,

$$u=\frac{d\phi}{dr}.\frac{x}{r}, \quad v=\frac{d\phi}{dr}.\frac{y}{r}, \quad w=\frac{d\phi}{dr}.\frac{z}{r},$$

values, which prove that the velocity is directed to or from the

origin of co-ordinates and is equal to $\frac{d\phi}{dr}$. The integral of $\frac{d^2.r\phi}{rdr^2}=0$, which is, $\phi=f(t)+\frac{F(t)}{r}$, determines the velocity $\frac{d\phi}{dr}$ to be $\frac{-F(t)}{r^2}$, and thus gives the law according to which it varies at different distances from the point to or from which the motion tends. This law may be verified by conceiving a small spherical ball, capable of expansion, to be placed concentric with a spherical fluid mass, inclosed in an envelope also capable of expansion. By the expansion of the ball, the particles will be moved through spaces which vary inversely as the squares of the distances from the centre. The supposition that ϕ is a function of r and t, does not necessarily restrict the application of the preceding integral to a particular case; for the law of motion it discloses must obtain wherever the parts of the fluid act on each other. Moreover, if the equation, $d\phi=udx+vdy+wdz$, has the meaning above assigned to it, at one point, it must have the same at every point; and if it be applicable to every point at one instant, it will, as Lagrange has proved, be applicable to every point at every instant. We may consider, therefore, the preceding integral to have been obtained on the supposition that the origin and direction of co-ordinates were arbitrary, and consequently to be applicable to every point in motion*. It is necessary to suppose

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* Similar reasoning is applicable to that integral of

$$\frac{d^2\phi}{dt^2}=a^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right)$$

which is obtained by supposing ϕ to be a function of r and t. I have not clearly stated in Art. 14, of my Paper on the Small Vibrations of Elastic Fluids, (*Cam. Phil. Trans.* Vol. III. Part. I.) upon what principle this integral may be considered general. It is general, as regarding the mode of action of the parts of the fluid on each other. M. Poisson's integral of the same equation (*Mem. Acad. Scien.* 1818.) is general in a different sense;—in regard to its application to any proposed instance.

that ϕ is a function of r and t to obtain this integral, only because it is not possible, as it seems, to find the complete integral of equation (2). Happily in the case in which the motion is in space of two dimensions, the complete integral can be found, and we are able to shew that the same result is arrived at, whether we suppose ϕ to be a function of r and t, or determine the forms of the arbitrary functions in the complete integral, on the hypothesis that the origin and direction of co-ordinates are not fixed. This Proposition, which is important to the present theory, I have proved in the Annals of Philosophy, for *August* 1829.

2. The integral of (2) obtained above, seems to be that which is really useful for the solution of any proposed question; and it may be questioned, if the complete integral could be obtained, whether it would be serviceable in any other way than in conducting to this. For, let us now fix the origin and direction of the axes of co-ordinates in space, and let α, β, γ, be the co-ordinates of the point towards which or from which the motion at the point whose co-ordinates are x, y, z, tends. Then

$$\frac{d\phi}{dr} = \frac{-F(t)}{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2};$$

$$\text{also } dr = \frac{x-\alpha}{r}\,dx + \frac{y-\beta}{r}\,dy + \frac{z-\gamma}{r}\,dz,$$

$$\text{and } d\phi = \frac{-F(t)}{r^2}\left\{\frac{x-\alpha}{r}\,dx + \frac{y-\beta}{r}\,dy + \frac{z-\gamma}{r}\,dz\right\}.$$

a complete differential of a function of x, y, z, whenever α, β, γ, may be considered constant while x, y, z, vary in an indefinitely small degree. From what has been said above, this will always be the case where the parts of the fluid move *inter se*, and change their relative positions: but when the fluid moves in

such a manner that it may be considered solid, ϕ has no existence, and the pressure is determined by

$$p = \int (X dx + Y dy + Z dz),$$

the forces, X, Y, Z, including those arising from rotation.

We have then for the solution of any proposed problem, the equations,

$$p = V - \frac{d\phi}{dt} - \tfrac{1}{2}\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) \qquad (A),$$

$$\phi = f(t) + \frac{F(t)}{\sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}} = f(t) + \frac{F(t)}{r} \qquad (B);$$

$$\frac{d\phi}{dr} = \frac{-F(t)}{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2} = -\frac{F(t)}{r^2} \qquad (C).$$

The expression $-\frac{F(t)}{r^2}$ for the velocity, is to be taken with respect only to a portion of the fluid, for which α, β, γ, and the form of F remain the same, while r varies. This portion will in general be elementary; and the expression above will have to the general expression for the velocity, which must be a function of x, y, z, and t, a relation analogous to that between the two expressions for the same variable, derived from the general and the particular solutions of a common differential equation. The complete integral of (2), supposing it obtained, would shew, as it must contain arbitrary functions, that no necessary connexion exists between the velocity of one elementary portion and that of another contiguous to it, but such only as we choose to impose by vessels, pipes, or other means. Hence the form and value of $F(t)$ may change at a given instant from one portion to another; they may also change in the same portion from one instant to another: and α, β, γ, may change in like manner. The equations (A), (B), (C), are con-

sequently applicable to fluid contained in any irregular vessel and moving in any manner. The form and value of $F(t)$, and the values of α, β, γ, for each point, at a given instant, must be determined from the shape of the vessel, the velocity and direction of the velocity at the given instant at parts of the fluid which are free, and from the law just proved of the communication of velocity according to the inverse square of the distance. For on these data alone depend the quantity and direction of the velocity at any point in the interior of the fluid mass. And as, when motion takes place, there must, at least, be two parts of the fluid which are free, and where the pressure is known, the equation (A) is proper for determining the motion at these parts at any instant, when the motion is given at a given instant. This equation is to be made use of, according to the following principles. As it has been shewn that ϕ is generally a function of r and t, which retains its form, at a given instant, when r varies in an indefinitely small degree,

$$\text{let } \phi = \chi(r, t); \quad \text{then } \phi' = \chi(r', t);$$
$$\phi' - \phi = \chi'(r, t)\,(r' - r) = \omega(r' - r),$$

supposing the velocity, which is $\dfrac{d\phi}{dr}$, to be represented by ω.

$$\text{Hence } \frac{d\phi'}{dt} - \frac{d\phi}{dt} = \frac{d\omega}{dt}(r' - r).$$

Here $r' - r$ may be considered the increment of a line s, drawn from a point at which the pressure and the direction of the velocity are known, continually in the direction of the motion of the particles through which it passes.

$$\text{Then } d.\frac{d\phi}{dt} = \frac{d\omega}{dt}ds; \quad \frac{d\phi}{dt} = \int\frac{d\omega}{dt}ds + f(t).$$

$$\text{Consequently, } p = V - \int\frac{d\omega}{dt}ds - \frac{\omega^2}{2} - f(t) \qquad (D).$$

The differential coefficient $\frac{d\omega}{dt}$ is the ratio of the increment of velocity to the increment of time, considered independently of the change of space; the integral indicated above is to be performed in reference to s, t being constant. The preceding reasoning shews that both the line s and the function $\frac{d\omega}{dt}$ may be discontinuous. It is not easy to apply the equation (D), on account of the difficulty of determining the values of α, β, and γ, which fix the position of the line s, and of ascertaining the velocity at every point of this line, in terms of the velocity at the point where the pressure is known. When, however this has been effected, we may obtain, by means of the known pressures at two points of this line, an equation proper for determining the velocity at one of the points at any time, when the velocity is given at a given time; and by inference the velocity at every point of the line. Thus the problem will be completely solved. The process here indicated, is that which has in fact been adopted in the problems which have admitted of solution; but in most cases the mathematical difficulties are too great to be overcome. I proceed to take one or two simple instances.

(1). Conceive the disturbance to be made in any mass of fluid, acted upon by no forces, by a spherical body expanding or contracting according to a given law, and in the same degree in all directions from the centre.

$$\text{Then } \phi = f(t) + \frac{F(t)}{r}; \quad \frac{d\phi}{dr} = \omega = -\frac{F(t)}{r^2};$$

$$\frac{d\phi}{dt} = f'(t) + \frac{F'(t)}{r}.$$

$$\text{Hence, } p = -\frac{F'(t)}{r} - \frac{\omega^2}{2} - f'(t).$$

If the expanding body cause the particles in contact with it to move with a velocity varying inversely as the square of the distance from the centre, $F(t) =$ a constant, $F'(t) = 0$; and if at the same time P be the pressure where r is infinitely great,

$$p = P - \frac{\omega^2}{2}.$$

The pressure is consequently less as the distance from the centre is less. If the expanding body *begin* to move with the velocity $\sqrt{2P}$, and go on moving according to the supposed law, the pressure of the fluid in contact with it, commencing with nothing, will go on increasing: if the initial velocity be greater than $\sqrt{2P}$, the fluid will be made to fly off from the expanding body, and a vacant space will be produced.

(2). To take an example of the equation,

$$p = V - \int \frac{d\omega}{dt}\, ds - \frac{\omega^2}{2} - f'(t),$$

suppose the ventricle of the heart to contract according to a law of velocity indicated by $\sin \dfrac{\pi a t}{\lambda}$: then, as the contractions are small, the velocity with which the fluid enters the great *aorta* will follow the same law, and may be equal to $m \sin \dfrac{\pi a t}{\lambda}$. If we suppose the arteries to be rigid, and the velocity in them at each point to remain the same in direction while it alters in quantity, that is, if the values of α, β, γ, be supposed independent of the time, which must be very approximately the case, we shall have,

$$\omega = \phi(s) . m \sin \frac{\pi a t}{\lambda}, \text{ and } \frac{d\omega}{dt} = \phi(s) . \frac{\pi a m}{\lambda} \cos \frac{\pi a t}{\lambda}.$$

Hence, as $V = gz$, $p = gz - \frac{\pi am}{\lambda} \cos \frac{\pi at}{\lambda} \int \phi(s)\,ds - \frac{\omega^2}{2} - f'(t)$.

Let z be measured from a level where the pressure arising from gravity is 0; and let $P_1 + gh$ be the pressure where $z = h$, $\omega = \omega_1$, $\int \phi(s)\,ds = -K_1 l$, K_1 being a quantity of no linear dimensions:

$$\text{Then } P_1 + gh = gh + K_1 l \,.\, \frac{\pi am}{\lambda} \,.\, \cos \frac{\pi at}{\lambda} - \frac{\omega_1^2}{2} - f'(t),$$

$$\text{and } p - P_1 = gz - \frac{\pi am}{\lambda} \cos \frac{\pi at}{\lambda} \{K_1 l + \int \phi(s)\,ds\} - \frac{\omega^2 - \omega_1^2}{2}.$$

At the entrance of the great aorta let

$$z = h', \quad \int \phi(s)\,ds = -K \,.\, l, \quad p = P, \text{ and } \omega = \varpi.$$

$$\text{Then } P = P_1 + gh' + \frac{\pi am}{\lambda} \cos \frac{\pi at}{\lambda} (K - K_1)\, l - \frac{\varpi^2 - \omega_1^2}{2}.$$

As it is known that the transverse section of the mean channel of the blood is greater, the greater the distance from the heart, K_1 will be less than K, and ω_1 less than ϖ; and the rather so, as K_1 and ω_1 refer to a point more distant from the heart. The value of P will be least when ϖ is greatest. It is possible that the right hand side of the equation above may become negative, in which case the blood might be entirely expelled from the ventricle, as some anatomists suppose it to be. The value of P will be greatest when $\varpi = 0$, and consequently $\omega_1 = 0$. In this case

$$P = (P_1 - K_1 l \,.\, \frac{\pi am}{\lambda} + gh') + Kl \,.\, \frac{\pi am}{\lambda}.$$

The quantity in brackets is the least pressure that is requisite for carrying on the pulsations: let it be equal to $\Pi + gh'$. As it

contains h', it depends on the position of the animal. The total pressure on the surface of the ventricle, when at a maximum, varies as $l^2 P$; the number of pulsations in a given time is $\frac{a}{2\lambda}$.

$$\text{Hence, as } l^2P - l^2(\Pi + gh') \propto l^3 . m . \frac{a}{\lambda},$$

the excess of active force above the least that is requisite for carrying on the pulsations, varies in the same animal, as the greatest velocity impressed on the fluid, multiplied by the number of pulsations in a given time, and in animals of the same construction, as the product of the mass of the fluid, the maximum velocity, and the number of pulsations in a given time.

3. There is one case of frequent occurrence, in which the equation (D) may be readily applied; viz. when the motion has arrived at a uniform state, so that the velocity of every particle passing through the same point, is the same in quantity and direction. In this case ω is independent of the time, $\frac{d\omega}{dt} = 0$,

$$\text{and } p = V - \frac{\omega^2}{2} + \text{constant*}.$$

This equation is applicable to the issuing of water, retained at a constant elevation, in *any* vessel, through any small orifice, or adjutage fitted to the orifice. Let α be the ratio of the velo-

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* This and the corresponding equation for elastic fluids, (Art. 9.) may be more simply deduced by combining D'Alembert's principle with a known property of the equilibrium of fluids: viz. that the pressure at any point of a fluid mass, is obtained by integrating in regard to any line whatever, drawn from the point to the free parts of the fluid. In this way they have been deduced by Mr. Moseley, in his Treatise on Hydrodynamics lately published; and he has been the first to observe them. Considering the number of phenomena, at first sight paradoxical, which these equations serve to explain, few things more valuable in the theory of fluid motion have been discovered since the time of D'Alembert.

city at the upper surface to the velocity of the issuing fluid, where the pressure $= P$ the atmospheric pressure; and let z be measured vertically from the surface. The equation for this instance is,

$$p - P = gz - \frac{\omega^2}{2}(1 - a^2)\ldots\ldots\ldots\ldots\ldots\ldots(f).$$

Several inferences may be drawn from this equation, and compared with experiment.

(1) The velocity at *every point* of the issuing stream, which is in immediate contact with the air, is very nearly the same, and consequently, by reason of the contraction which experiment makes known, greater than the velocity at any point in the interior of the part of the stream between the orifice and *vena contracta*. Hence in this part, the pressure will every where be greater than the atmospheric pressure. Just at the contracted vein there will be a transverse section of the stream, at every point of which the velocity will be the same, and the pressure will be P, because the section has no tendency to increase or diminish; and if h be its depth below the constant surface, the velocity

$$= \sqrt{\frac{2gh}{1 - a^2}}.$$

(2) Conceive the stream to descend vertically, and a tube to be fitted to the orifice, its upper part having the form the vein of fluid assumes on entering the air, as far as the *vena contracta*, and its lower part being cylindrical. Let h be the distance of the lower extremity from the constant level of the fluid, h' that of the *vena contracta*. Then it will be seen that the expenditure is increased by the adjutage, by

$$\sqrt{\frac{2g}{1 - a^2}}.(\sqrt{h} - \sqrt{h'}),$$

and the pressure at the contracted vein is less than the atmospheric pressure by $g(h - h')$, the weight of the column of fluid in the cylindrical part of the tube. This agrees with Prop. IV. of Venturi, if we leave out of consideration the effects of the inequalities of the tube.

(3) Suppose a cylindrical tube to be fitted to a circular orifice, and to be placed with its axis horizontal. When the fluid fills the tube, the velocity of issuing into the air will be $\sqrt{\frac{2gh}{1-a^2}}$, and the expenditure will consequently be increased by the adjutage in the ratio of the area of the orifice, to the transverse section of the stream at the *vena contracta*. This ratio is found by experience to be $\frac{25}{16}$. Venturi relates an experiment, in which the time of expending a given quantity of water through the orifice was 41″, and through the tube 31″. By theory the latter interval is 26¼″. This difference is shewn by experiments to be principally owing to the retardation caused by the inequalities of the tube, and the eddies and irregularities of the motion within it.

(4) Conceive the tube to be indented at the *vena contracta*, so that the minimum transverse section may be the same as the section of the contracted vein in air, and let it be required to find the pressure at the minimum section. If p = this pressure, it will be found, on taking the experimental value of the velocity of issuing in the example just adduced, that

$$P - p = \left\{\left(\frac{41}{31}\right)^2 - 1\right\} gh = .7 \cdot gh \text{ nearly.}$$

In an experiment in which h was 32.5 inches Venturi found $P - p$ to be the weight of 24 inches. Theory gives 22¾, which is as near as can be expected.

(5) A fourth kind of adjutage employed by Venturi, was one which converged to the *vena contracta*, and then diverged from it. The equation (f) shews that, as the velocity will decrease in passing from the minimum section to the mouth of the tube, the pressure must increase: and this experience confirms, and gives a rate of increase nearly agreeing with what would result from the theory, on the supposition that the velocity varies inversely as the transverse section of the tube. In this example the stream is diverging as it leaves the mouth, and the velocity of that portion of it which is in immediate contact with the air, is very nearly the same, and must consequently be less than the velocity at every point in the interior of the stream, within a short distance from the aperture. As there must be a section a little distant from the aperture, at which the stream ceases to be divergent, here the velocity at every point will be the same, and the pressure equal to that of the atmosphere. This velocity will be $\sqrt{\frac{2gh}{1-a^2}}$; and as the maximum section will be larger than the aperture, this will account for a greater expenditure from a conical diverging tube, than from a cylindrical of the same aperture:—which Venturi found to be the case.

(6) The last example will serve to explain the phenomenon observed by M. Hachette, of the attraction of a disc, opposed to a stream of water, issuing from an opening in a plane surface. When the disc, which we suppose to be circular as well as the opening, is placed sufficiently near the surface for the fluid to fill the interval between them, the stream may be divided into any number of equal portions, similarly situated in regard to the centre of the disc, each of which is divergent in passing towards the edge. And as the fluid on entering the air, is subject to the atmospheric pressure, the pressure of that part which is

in contact with the disc, will, according to the theory, be less than that of the atmosphere. This will not be the case just about the centre, for there, from the manner of flowing, the fluid must be in some degree stationary.

The preceding problems suffice to exemplify the nature of the integral we set out with, and to shew the manner of making use of it: I proceed to the consideration of elastic fluids.

II. *Compressible Fluids.*

4. The general equations relating to the motion of compressible fluids, in which the pressure varies as the density, are,

$$a^2 \text{ hyp. log. } \rho = V - \frac{d\phi}{dt} - \frac{1}{2}\left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right) \ldots\ldots\ldots\ldots (m),$$

$$\begin{aligned} 0 = a^2 \left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right) - \frac{d^2\phi}{dt^2} \\ + \frac{dV}{dx}.\frac{d\phi}{dx} + \frac{dV}{dy}.\frac{d\phi}{dy} + \frac{dV}{dz}.\frac{d\phi}{dz}, \\ - 2\frac{d\phi}{dx}.\frac{d^2\phi}{dxdt} - 2\frac{d\phi}{dy}.\frac{d^2\phi}{dydt} - 2\frac{d\phi}{dz}.\frac{d^2\phi}{dzdt} \quad (n), \\ - \frac{d\phi^2}{dx^2}.\frac{d^2\phi}{dx^2} - \frac{d\phi^2}{dy^2}.\frac{d^2\phi}{dy^2} - \frac{d\phi^2}{dz^2}.\frac{d^2\phi}{dz^2}, \\ - 2\frac{d\phi}{dx}.\frac{d\phi}{dy}.\frac{d^2\phi}{dxdy} - 2\frac{d\phi}{dx}.\frac{d\phi}{dz}.\frac{d^2\phi}{dxdz} - 2\frac{d\phi}{dy}.\frac{d\phi}{dz}.\frac{d^2\phi}{dydz} \end{aligned}$$

(Lagrange, *Mec. Anal.* Part II. Sect. 12.)

Here the pressure p is equal to $a^2\rho$, ρ being the density. Also as before

$$dV = Xdx + Ydy + Zdz, \text{ and } d\phi = udx + vdy + wdz.$$

If we put $r^2 = x^2 + y^2 + z^2$, and suppose ϕ to be a function of r and t, we shall find that the equation (n) will reduce itself to

$$0 = a^2\frac{d^2\phi}{dr^2} - \frac{d^2\phi}{dt^2} - 2\frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt} - \frac{d\phi^2}{dr^2}\cdot\frac{d^2\phi}{dr^2}$$
$$+ \left(\frac{2a^2}{r} + \frac{Xx}{r} + \frac{Yy}{r} + \frac{Zz}{r}\right)\frac{d\phi}{dr}. \qquad (p).$$

This equation does not accord in giving ϕ a function of r and t, unless

$$\frac{Xx}{r} + \frac{Yy}{r} + \frac{Zz}{r},$$

which is the part of the impressed force resolved in the direction of r, be a function of r and t. In consequence of the supposition respecting ϕ, and the equation

$$d\phi = udx + vdy + wdz,$$

this direction is that of the motion. As, however, this resolved force may be considered a function of r and t, the form of which is constant for an indefinitely small portion of the fluid, the equation (p) shews that for every elementary portion, ϕ is a function of r and t, and consequently that the motion at every point is directed to a fixed or moveable centre, and varies according to a law to be determined by the integration of this equation. This law is not, as in incompressible fluids, independent of the impressed forces.

5. Let us suppose the force to act in the direction of r and to be P, and r to be infinitely great. Then (p) becomes,

$$0 = a^2\frac{d^2\phi}{dr^2} - \frac{d^2\phi}{dt^2} - 2\frac{d\phi}{dr}\cdot\frac{d^2\phi}{drdt} - \frac{d\phi^2}{dr^2}\cdot\frac{d^2\phi}{dr^2} + P\frac{d\phi}{dr},$$

and is identical in form with the equation that would be obtained from (n), by supposing this equation to contain only one of the variables, for instance x: and this plainly should be the case.

$$\text{Let } r = \text{constant} + s, \text{ and } \frac{d\phi}{dr} \text{ or } \frac{d\phi}{ds} = \omega.$$

We have then,

$$0 = \frac{d^2\phi}{ds^2} - \frac{2\omega}{a^2-\omega^2} \cdot \frac{d^2\phi}{ds\,dt} - \frac{1}{a^2-\omega^2} \cdot \frac{d^2\phi}{dt^2} + \frac{P\omega}{a^2-\omega^2} \qquad (q).$$

I will, in the first instance, consider this equation. By treating it according to the method of Monge, the following two systems of equations will arise:

$$\left.\begin{aligned} ds - (a+\omega)\,dt = 0 \\ (a-\omega)\,d\omega - d.\frac{d\phi}{dt} + \frac{P\omega ds}{a+\omega} = 0 \end{aligned}\right\} (r).$$

$$\left.\begin{aligned} ds + (a-\omega)\,dt = 0 \\ (a+\omega)\,d\omega + d.\frac{d\phi}{dt} + \frac{P\omega ds}{a-\omega} = 0 \end{aligned}\right\} (s),$$

and it is observable that one of these systems is convertible into the other by changing the sign of a. By integrating the equations (r), putting for dt its value $\frac{ds}{a+\omega}$,

$$s - at - \int \frac{\omega ds}{a+\omega} = c,$$

$$a\omega - \frac{\omega^2}{2} - \frac{d\phi}{dt} + \int \frac{P\omega ds}{a+\omega} = c'.$$

These two equations are between s, t, c, and c'; and $c'=f(c)$; therefore any one of the four quantities is a function of two of them; and as ω is a function of s and t, we may put,

$$\omega = \chi\,(c, s).$$

Hence, $a\omega - \frac{\omega^2}{2} - \frac{d\phi}{dt} = f\left\{s - at - \int \frac{\chi(c, s)\, ds}{a + \chi(c, s)}\right\} - \int \frac{P\chi(c, s)\, ds}{a + \chi(c, s)}$.

So from the equations (s),

$$a\omega + \frac{\omega^2}{2} + \frac{d\phi}{dt} = F\left\{s + at + \int \frac{\chi_{,}(c_{,}, s)\, ds}{a - \chi_{,}(c_{,}, s)}\right\} - \int \frac{P\chi_{,}(c_{,}, s)\, ds}{a - \chi_{,}(c_{,}, s)}.$$

We are able to obtain particular integrals from these equations, either when $P = 0$, or is constant. First, suppose no force to act. Then,

$$2a\omega = f\left\{s - at - \int \frac{\chi(c, s)\, ds}{a + \chi(c, s)}\right\} + F\left\{s + at + \int \frac{\chi_{,}(c_{,}, s)\, ds}{a - \chi_{,}(c_{,}, s)}\right\};$$

and, because a^2 hyp. log. $\rho = -\frac{d\phi}{dt} - \frac{\omega^2}{2}$,

$$2a^2 \text{ hyp. log. } \rho = f\left\{s - at - \int \frac{\chi(c, s)\, ds}{a + \chi(c, s)}\right\} - F\left\{s + at + \int \frac{\chi_{,}(c_{,}, s)\, ds}{a - \chi_{,}(c_{,}, s)}\right\}.$$

According to a foregoing remark, the quantities of which f and F are respectively functions, are convertible into each other by changing the sign of a. Now we know, that when the motions are small, a change of the sign of a has reference to a change in the direction of propagation; and that propagation may obtain either in a single direction, or simultaneously in opposite directions. Let us endeavour, therefore, to ascertain whether the preceding equations will satisfy (q), when one of the arbitrary functions, F for instance, is supposed to disappear. In this case,

$$2a\omega = 2a^2 \text{ hyp. log. } \rho = f(c).$$

Hence $\omega = \frac{f(c)}{2a}$; and $\chi(c, s)$ is therefore a function of c only: so that

$$\int \frac{\chi(c, s)\, ds}{a + \chi(c, s)} = \frac{\frac{f(c)}{2a} \cdot s}{a + \frac{f(c)}{2a}} = \frac{\omega s}{a + \omega}.$$

Hence, $2a\omega = 2a^2$ hyp. log. $\rho = f\left(s - at - \frac{\omega s}{a+\omega}\right) = f\left(\frac{as}{a+\omega} - at\right)$ (1).

So by supposing the function f to disappear, we should find that,

$$2a\omega = -2a^2 \text{ hyp. log. } \rho = -F\left(\frac{as}{a-\omega} + at\right) \qquad (2).$$

These two particular integrals will be found by trial to satisfy (q): consequently each indicates separately a motion which is possible. Let us suppose in (1) s and t to vary in such a manner that ω does not alter. The consequent value, $a+\omega$, of $\frac{ds}{dt}$, is the velocity of propagation *together with* the velocity at the point whose abscissa is s. Hence the velocity of propagation is a, the same quantity for every point in motion. By considering in the same manner the equations (2), the velocity of propagation will be found to be $-a$, the negative sign indicating the contrary direction. It thus appears that, when an impression is made on the fluid in a single direction, whatever be the magnitude of the motions, the velocity of propagation at every point in motion is precisely a. This remarkable result, which discontinuous functions have hitherto concealed from the eyes of mathematicians, may be confirmed by the following reasoning. Suppose the line of fluid we are considering, to be divided into an unlimited number of equal masses, indefinitely small, and call three of them taken in succession, α, β, γ; α being that which is nearest the origin of abscissæ. Let $z' =$ the distance between the centres of gravity of α and β; z, that between the centres of gravity of β and γ; ω', ω, the velocities respectively of the centres of gravity of β and γ: then $\omega' - \omega$ is their relative velocity. Suppose at the end of an interval τ reckoned from the instant at which the distances were z' and z, that the latter becomes z': the relative velocity $\omega' - \omega$ will be constant during

this interval, if we omit the consideration of small quantities of an order which may be neglected. It will follow that

$$(\omega' - \omega)\,\tau = z - z';$$

for ω' is greater or less than ω, according as z is greater or less than z'. But as the densities must vary inversely as the distances between the centers of gravity of the equal small masses, if ρ', ρ be the densities corresponding to z', z, respectively, $\frac{z'}{z} = \frac{\rho}{\rho'}$. Hence

$$\omega' - \omega = \frac{z}{\tau}\,\frac{\rho' - \rho}{\rho'}.$$

Now $\omega' - \omega$ is the variation of velocity at a given instant between two points indefinitely near each other, and $\rho' - \rho$ is a like variation of the density. Therefore $\frac{d\omega}{ds} = \frac{z}{\tau}\,\frac{d\rho}{\rho ds}$. As $\frac{z}{\tau}$ is the ratio of a linear dimension of a given mass of the fluid, to the time during which the state of density at one extremity of this line passes into the state at the other extremity, and as it is independent of the absolute velocity of the mass, it must be accurately the velocity of propagation. This quantity, $\frac{z}{\tau}$, in general will be a function of the abscissa and the time; but if it be constant and equal to a,

$$\frac{d\omega}{ds} = a\,.\,\frac{d\rho}{\rho ds}, \quad \omega = a \text{ hyp. log. } \rho + \phi(t);$$

$$\text{and } \phi(t) = 0, \text{ if } \omega = 0 \text{ where } \rho = 1.$$

This result accords with what is said above. It should be observed that the preceding reasoning is altogether independent of the constitution of the fluid, and that the equation

$$\frac{d\omega}{ds} = F(s, t)\,\frac{d\rho}{\rho ds},$$

is of very general application.

Again, as ω may be considered a function of c and t, or of c_1 and t, we may also have,

$$2a\omega = f\{s - at - \int\psi(c, t)\,dt\} + F\{s + at - \int\psi_1(c_1, t)\,dt\},$$

$$2a^2 \text{ hyp. log. } \rho = f\{s - at - \int\psi(c, t)\,dt\} - F\{s + at - \int\psi_1(c_1, t)\,dt\};$$

from which, by reasoning as before, may be obtained the two particular integrals,

$$2a\omega = \quad 2a^2 \text{ hyp. log. } \rho = \quad f(s - at - \omega t) \qquad (3),$$

$$2a\omega = -2a^2 \text{ hyp. log. } \rho = -F(s + at - \omega t) \qquad (4).$$

These give the same velocity of propagation as the integrals (1) and (2).

The integrals (3) and (4) were first obtained by M. Poisson, (*Journal de l'Ecole Polytechnique*, Cah. XIV.) and he infers from the consideration of discontinuous functions, that one of them relates to motion on the positive side of the origin of s, the other to motion on the negative. But this inference is made by employing the arbitrary functions in a manner, which their nature does not admit of. The existence of arbitrary functions in the integral is the proper proof of the discontinuity of the motion; for were the motion necessarily continuous, the value of ω would be given by a determinate function. The *arbitrary* character of the functions has reference to the mode of disturbance, which may be any we *please*, either obeying or not the law of continuity. Those properties of the motion which are independent of the manner of disturbance, must be ascertained by reasoning independent of the arbitrary nature of the functions. By reasoning in this way, we have shewn that one of the preceding solutions belongs to propagation in the positive direction, the other to propagation in the contrary direction, whether on the positive or negative side of the origin of s.

Also we can now shew why a disturbance, independently of its particular nature, produces propagations in opposite directions from the place of its application. For either of the equations

$$\omega = a \text{ hyp. log } \rho, \quad \omega = -a \text{ hyp. log. } \rho,$$

proves that where the fluid is condensed, the velocity is in the direction of propagation; where it is rarefied, in the opposite direction. Now every disturbance will condense the fluid in one part, and rarefy it as much at another, but will impress motion at all parts in the direction in which the impression is given; therefore the same disturbance will cause propagations in opposite directions.

6. Recurring now to the complete integral, we may infer from it, that two propagations will obtain simultaneously in opposite directions. It is not possible, as it seems, to obtain this integral in exact terms: let us suppose that,

$$2a\omega = f(s - at - \omega_1 t) - F(s + at - \omega_2 t),$$
$$2a^2 \text{ hyp. log. } \rho = f(s - at - \omega_1 t) + F(s + at - \omega_2 t);$$

in which ω_1, ω_2, are unknown functions of s and t. If the two opposite propagations be exactly alike, there will be one point at least where the velocities due to them respectively will be equal and opposite, and where consequently the resulting velocity will be nothing whatever be the time. Now at this point ω_1 will be equal to $-\omega_2$, whatever these functions may be, because we may presume that the value of $\frac{ds}{dt}$, obtained by differentiating $s - at - \omega_1 t = c$, on the supposition that ω_1 is constant, will differ only in sign from that which is obtained from $s + at - \omega_2 t = c_1$, on a like supposition with respect to ω_2. Let therefore $m =$ the distance of the point of rest from the origin of s,

$$\text{and } (a + \omega_1)t, \text{ or } (a - \omega_2)t = z.$$

Then $f(m-z) - F(m+z) = 0$, whatever be z.

Hence $f(m) - F(m) - \{f'(m) + F'(m)\} z + \{f''(m) - F''(m)\} \frac{z^2}{2} - \&c. = 0$,

independently of any relation between m and z. Therefore

$$\begin{array}{ll} f\ (m) - F\ (m) = 0, & f'\ (m) + F'\ (m) = 0, \\ f''\ (m) - F''(m) = 0, & f'''\ (m) + F'''(m) = 0, \\ f^{\text{iv}}\ (m) - F^{\text{iv}}(m) = 0, & f^{\text{v}}\ (m) + F^{\text{v}}\ (m) = 0, \\ \&c. & \&c. \end{array}$$

These equations are to be satisfied so that m may have in all the same arbitrary value. If we suppose f and F identical, we must have

$$f'(m) = \pm f'''(m) = \pm f^{\text{v}}(m) = \&c. = 0.$$

These conditions are satisfied by the equation

$$f'(m) = \sin\ (m + \gamma),$$

when $m = -\gamma$ an arbitrary quantity. Again, if f and $-F$ be identical, we must have

$$f(m) = \pm f''(m) = \pm f^{\text{iv}}(m) = \&c. = 0.$$

The required conditions are satisfied by the equation

$$f(m) = \sin\ (m + \gamma'),\ \text{when}\ \ m = -\gamma'.$$

As $f'(m) = \cos\ (m + \gamma')$, we may have

$$\mp\ (m + \gamma') = \pm \left\{\frac{n\pi}{2} - (m + \gamma)\right\},\ \ n \text{ being odd.}$$

$$\text{Hence}\ \ \gamma - \gamma' = \frac{n\pi}{2};$$

which shews that the arbitrary values of m, which result from the two hypotheses respecting f and F, are related to each other, if $f'(m)$ be the same in the two cases. These two are the only hypotheses on which all the equations above can be

satisfied by the same value of m: the two forms of the arbitrary function they give are virtually the same. The form thus determined, evidently possesses the properties indicated by the equations,

$$f(l-z) - f(l+z) = 0,$$
$$f(l-z) + f(l+z) = 0.$$

The same properties are possessed by the function

$$\sin l + m \sin 3l + m' \sin 5l + \&c.,$$

which, on account of the unlimited number of terms, fulfils the required conditions in the most general manner, and is inclusive of every function that fulfils them. But the primary form of the arbitrary function is $\sin l$, and every other form points to motion compounded of the motion indicated by this. As we have arrived at this form by reasoning upon the arbitrary functions, on the supposition that the origins of s and t are arbitrary, it must refer to every elementary portion of the fluid, and indicate the mode of action of the parts on each other.

In general we shall have,

$$\omega = a \text{ hyp. log. } \rho = m \sin \frac{\pi}{\lambda} . (s - at - \omega t);$$

and by supposing m and λ to vary at a given instant from one point to another, either continuously, or in a manner not subject to the law of continuity, these equations will accommodate themselves to any mode of vibration we choose to impress on the fluid. When m and λ are constant whatever be s and t, the vibrations are of a particular kind, which may be called primary, and are generated by the action of the parts of the fluid on each other. In this case, the velocity and density at a given instant, have the same values at points separated by the constant interval 2λ. Also at a given point, the velocity and

density will return to the same values after equal intervals of time; and it would seem from the equation above, that the interval for the velocity ω is $\frac{2\lambda}{a+\omega}$; whereas it should be $\frac{2\lambda}{a}$ since the propagation is uniform. But it is to be observed that the equations of the motion are also satisfied by

$$\omega = a \text{ hyp. log. } \rho = m \sin \frac{\pi}{\lambda}\left(\frac{sa}{a+\omega} - at\right),$$

from which it may be inferred that at a given point, the instants at which the same velocity recurs are separated by the constant interval of time $\frac{2\lambda}{a}$. The two integrals compared together, shew that in the latter λ may either be constant, or may vary inversely as $a+\omega$: that it is constant, when the variation of velocity and density at a given point is to be determined, and varies inversely as $a+\omega$, when the variation of velocity and density from one point to another, is to be determined at a given instant. A similar observation may be made with respect to the other integral.

I will here observe that the form of the arbitrary function found above, indicates the kind of vibration which Newton selected in his theory of sound, without giving any reason for his selection. He might have inferred from his law of the vibrations, that two propagations could obtain simultaneously in opposite directions, and appealed to experience for a confirmation of the truth of the inference. In the analytic theory the process of reasoning is in the reverse order; the possibility of the simultaneous propagations is first proved, and thence the law of the vibrations is deduced. This requires the solution of a functional equation, according to the principles we have exhibited.

7. Now let a constant force g act on the fluid. The equations (r), Art. 5. for this instance become,

$$s - at - \int \omega dt = c,$$

$$a\omega - \frac{\omega^2}{2} - \frac{d\phi}{dt} + g\int \omega dt = c'.$$

Hence $a\omega - \frac{\omega^2}{2} - \frac{d\phi}{dt} + g(s - at) = c' + gc = f_1(c) + gc = f(c)$.

So from the equations (s),

$$a\omega + \frac{\omega^2}{2} + \frac{d\phi}{dt} - g(s + at) = c'_1 = gc_1 = F_1(c_1) - gc_1 = F(c_1).$$

From these two equations, by reason of the equation

$$a^2 \text{ hyp. log. } \rho = gs - \frac{d\phi}{dt} - \frac{\omega^2}{2},$$

we obtain,

$$2a(\omega - gt) = f(c) + F(c_1),$$
$$2a^2 \text{ hyp. log. } \rho = f(c) - F(c_1).$$

Of these functions, f refers to propagation in the positive direction, F to propagation in the negative. To obtain a particular integral, suppose that $F(c_1) = 0$. Then

$$\omega - gt = a \text{ hyp. log. } \rho = \frac{f(c)}{2a}.$$

Hence $\int \omega dt = \frac{gt^2}{2} + \frac{f(c)}{2a}.t = \omega t - \frac{gt^2}{2}$;

or, because $d\omega = gdt$, $\int \omega dt = \frac{\omega^2}{2g}$. Hence finally,

$$\omega - gt = a \text{ hyp. log. } \rho = f\left(s - at - \omega t + \frac{gt^2}{2}\right);$$

or, $\omega - gt = a \text{ hyp. log. } \rho = f\left(s - at - \frac{\omega^2}{2g}\right)$.

Both these integrals will be found to satisfy the given equations. If in the first we make s and t vary so that ρ does not alter, we find $ds - adt - \omega dt - td\omega + gtdt = 0$. But as ρ does

not vary, $d\omega = gdt$; therefore $\frac{ds}{dt} = a + \omega$, and the velocity of propagation is exactly a. This result might have been deduced from the general formula, $\omega = a$ hyp. log. $\rho + \phi(t)$, in Art. 5. The same velocity of propagation will be found by employing the other integral.

8. I return now to the equation (p) Art. 5. Let P be the force obtained by resolving the impressed forces in the direction of r, and let it be either constant or a function of r. Then

$$\frac{d^2\phi}{dr^2} - \frac{2\omega}{a^2-\omega^2}\cdot\frac{d^2\phi}{drdt} - \frac{1}{a^2-\omega^2}\cdot\frac{d^2\phi}{dt^2} - \left(\frac{2a^2}{r} + P\right)\frac{\omega}{a^2-\omega^2} = 0 \quad (M),$$

$$a^2 \text{ hyp. log. } \rho - \int P dr + \frac{d\phi}{dt} + \frac{\omega^2}{2} = 0 \quad (N).$$

By treating (M) according to the method of Monge, the following two sets of equations are obtained:

$$\left.\begin{array}{r} dr - adt - \omega dt = 0 \\ ad\omega - \omega d\omega - d.\frac{d\phi}{dt} + \left(\frac{2a^2}{r} + P\right)\frac{\omega dr}{a+\omega} = 0 \end{array}\right\} \quad (\alpha),$$

$$\left.\begin{array}{r} dr + adt - \omega dt = 0 \\ ad\omega + \omega d\omega + \frac{d\phi}{dt} + \left(\frac{2a^2}{r} + P\right)\frac{\omega dr}{a-\omega} = 0 \end{array}\right\} \quad (\beta).$$

Now multiplying the first of the equations (α) by P, integrating the two, and adding the integrals, we shall have, because $\frac{dr}{a+\omega} = dt$,

$$a\omega - \frac{\omega^2}{2} - \frac{d\phi}{dt} + \int P dr + 2a^2 \int_{,} \frac{\omega dt}{r} - a\int_{,} P dt = c + c' = c + f_{1}(c) = f(c).$$

So from the equations (β), because $\frac{dr}{a-\omega} = -dt$,

$$a\omega + \frac{\omega^2}{2} + \frac{d\phi}{dt} - \int P dr - 2a^2 \int_{\circ} \frac{\omega dt}{r} - a\int_2 P dt = c_1 + c'_1 = c_1 + F_1(c_1) = F(c_1).$$

Hence $2a\omega + a\int_1 \left(\frac{2a\omega}{r} - P\right) dt - a\int_2 \left(\frac{2a\omega}{r} + P\right) dt = f(c) + F(c_1)$,

$$2a^2 \text{ hyp. log. } \rho + a\int_1 \left(\frac{2a\omega}{r} - P\right) dt + a\int_2 \left(\frac{2a\omega}{r} + P\right) dt = f(c) - F(c_1).$$

The integral $\int_1$ is to be performed on the supposition that c is constant, and $\int_2$ on the supposition that c_1 is constant. We cannot hope to effect these integrations; but it is worth while to attend to the case in which $F(c_1) = 0$. In this case

$$\omega - a \text{ hyp. log. } \rho - \int_2 \left(\frac{2a\omega}{r} + P\right) dt = 0; \qquad (H)$$

and if r be so great that the term $\frac{2a\omega}{r}$ may be neglected, we fall upon the same equation as would be obtained for propagation in a single (the positive) direction, by the consideration of motion in space of one dimension. In fact, the two systems of equations (α) and (β), which are convertible one into the other by changing the sign of a, imply that in space of three dimensions, two propagations may exist either simultaneously or separately, one directed towards, the other from, a centre.

It may be useful to observe that when the motions are small and $P = 0$, the law of the variation of the velocity very near the centre, is the same as in incompressible fluids. We may confirm this remark in the following manner. Let us suppose in (H) that $\omega = \frac{k}{r^2}$, as in the first example of Art. 2. Then,

$$\int \frac{2ak}{r^3} dt = 2a \int \frac{-k dr}{r^3 \left(a - \frac{k}{r^2}\right)} = -a \text{ hyp. log. } \frac{a-\omega}{a}.$$

$$\text{Hence } \omega - a \text{ hyp. log.} \frac{\rho a}{a - \omega} = 0;$$

from which, neglecting $\left(\frac{\omega}{a}\right)^3$, &c.,

$$a^2 \rho = a^2 - \frac{\omega^2}{2}, \text{ or } p = \Pi - \frac{\omega^2}{2},$$

just as in the above-mentioned example. The foregoing remark may also be verified by the integral of the equation,

$$\frac{d^2 . r\phi}{dt^2} = a^2 . \frac{d^2 . r\phi}{dr^2},$$

which is derived from (n) Art. 4, by taking only the two first terms of this equation, and supposing ϕ to be a function of r and t. If $\rho = 1 + s$, this integral gives, when s is small,

$$as = \frac{1}{r} \{F'(r - at) - f'(r + at)\},$$

$$\omega = \frac{1}{r} \{F'(r - at) + f'(r + at)\} - \frac{1}{r^2} \{F(r - at) + f(r + at)\} *.$$

Hence when ω is small and r very small,

$$as = 0, \text{ and } \omega = - \frac{F(r - at) + f(r + at)}{r^2},$$

nearly; and the arbitrary functions may be so assumed that ω shall be equal to $\frac{\chi(t)}{r^2}$.

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* In Art. 14 of the communication I made to the Society, on the Small Vibrations of Elastic Fluids, the term involving $\frac{1}{r^2}$ in the value of ω is by mistake written $\frac{\phi}{r^2}$, and the reason assigned for neglecting this term is inaccurate. It can only be neglected when r is great, compared to the distance to which the effect of the disturbance in the first instant extends, or is great compared to λ the breadth of an undulation. Hence also, the manner in

9. We have seen Art. 5. that ϕ is generally a function of r and t, the form of which at a given instant is constant when r is made to vary through a very small space; and that the velocity is $\frac{d\phi}{dr}$.

$$\text{Hence if } \phi = \phi(r, t), \quad \phi' - \phi = \phi(r', t) - \phi(r, t)$$

$$= \frac{d.\phi(r, t)}{dr}(r' - r) = \omega(r' - r):$$

$$\text{and } \frac{d\phi'}{dt} - \frac{d\phi}{dt} = \frac{d\omega}{dt}(r' - r).$$

Here $r' - r$ may be considered the increment of a line s, drawn from a fixed point, always in the direction of the motion of the particles through which it passes.

$$\text{Hence } d.\frac{d\phi}{dt} = \frac{d\omega}{dt}ds;$$

$$\frac{d\phi}{dt} = \int \frac{d\omega}{dt}ds + f(t).$$

$$\text{Consequently, } a^2 \text{ hyp. log. } \rho = V - \int \frac{d\omega}{dt}ds - \frac{\omega^2}{2} - f(t).$$

This equation is readily applicable, whenever the motion has attained to such a state, that the velocity of every particle passing through the same point, is the same in quantity and direction; for on this supposition $\frac{d\omega}{dt} = 0$ at every point, and in consequence,

$$a^2 \text{ hyp. log. } \rho = V - \frac{\omega^2}{2} - f(t) \qquad (K).$$

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in which a series of waves will act upon a small solid, such as a slender chord, is not correctly stated in Art. 20, of that paper: it may be shewn, however, that the action is the same in kind as what is there stated, but different in degree, and that the equation for determining the vibrations of the chord is of the proper form.

(1) To exemplify this equation, let us take the instance of air contained in any vessel, and driven from it through any small orifice into the atmosphere, by being made subject to a given pressure. Let Π be the given pressure, and let ω_1 be the velocity where the pressure has this value. Then, neglecting the effect of gravity,

$$a^2 \text{ hyp. log. } \frac{\Pi}{a^2} = -\frac{\omega_1^2}{2} - f(t);$$

$$\text{from which, } 2a^2 \text{ hyp. log. } \frac{p}{\Pi} = \omega_1^2 - \omega^2; \qquad (4),$$

hence the pressure is less as ω is greater.

This property will enable us to explain the phænomenon, lately exhibited before the Society by Mr. Willis, of the attraction of a flat plate, opposed to a stream of air issuing from an orifice in a plane surface. Supposing the plate to be a circular disc, the orifice also to be circular, and their centres to be at a small distance from each other in a straight line, at right angles to the planes of the surface and the plate, it is evident that the stream may be divided into as many equal portions as we please, having all the same motion, and related in the same manner to the centre of the disc. Also if a circle be described with the centre of the disc as centre, and radius equal to any length r, the same quantity of fluid must pass this circle in the same time, whatever be r.

$$\text{Hence } \omega\rho r = c, \text{ and } \rho = \frac{c}{\omega r}.$$

Substituting this value of ρ in the equation above, and neglecting ω_1, which in the experiment was very small,

$$\omega e^{-\frac{\omega^2}{2a^2}} = \frac{a^2 c}{\Pi r}.$$

$$\text{Also } \rho^2 \text{ hyp. log.} \frac{\Pi}{a^2\rho} = \frac{c^4}{2a^2r^2}.$$

Hence the law of the variations of ω and ρ remains the same, whatever be the size of the disc, if c and Π be the same. According to these formulæ ω is less and ρ is greater as r is greater. But it is to be observed, that the theory supposes every particle of the air at the same distance from the centre of the disc, to be moving with the same velocity. This will not be the case if the disc be at any considerable distance from the orifice, because the current, from the manner of its passage out of the orifice, must, when entering upon the disc, suffer contraction in the vertical sense, supposing the disc to be situated with its plane horizontal: and the equation

$$2a^2 \text{ hyp. log.} \frac{p}{\Pi} = -\omega^2,$$

shews that, as ω will in consequence increase, the pressure diminishes. This diminution will continue up to a certain distance, at which the velocity will be greatest and the pressure least, and where the *vena contracta* of the stream may be said to be situated. The distance of the *vena contracta* from the circumference of the orifice, will be greater as the distance of the disc from the orifice is greater. When this latter distance is .031 of an inch, or any quantity less than this, and the radius of the orifice is .188, experiment shews that the distance of the *vena contracta* from the centre of the disc is .28 nearly (See Tables IV. and V, in Mr. Willis's Paper, *Camb. Phil. Trans.* Vol. III. Part I.) Past the *vena contracta*, the preceding formulæ give the law of increase of density and diminution of velocity, up to the edge of a disc the radius of which was 1.2, with accuracy, provided the interval between the disc and orifice be not much less than

.031 of an inch. As this interval is diminished, it is found that the pressure attains a maximum before the stream leaves the disc. This must be owing to causes which the theory does not take into account: probably it is due to friction, the effect of which will become more sensible as the passage for the stream is contracted. Whatever be the force which occasions this condensation, let us call it P. Then by the equation (K),

$$a^2 \text{ hyp. log. } \frac{a^2\rho}{\Pi} = \int P dr - \frac{\omega^2}{2} = \int P dr - \frac{c^2}{2r^2\rho^2};$$

$$\text{from which, } \frac{a^2 d\rho}{\rho dr} = P + \frac{c^2}{r^3\rho^2} + \frac{c^2}{r^2\rho^3} \cdot \frac{d\rho}{dr};$$

$$\text{and if } \rho \text{ be a maximum, } P = -\frac{c^2}{r^3\rho^2} = -\frac{\omega^2}{r}.$$

This shews that the greater P is, the nearer the maximum pressure is to the centre; and the experiments confirm this result; for when the interval between the disc and orifice was diminished, the circle of maximum pressure was drawn towards the centre. Upon the whole then, there will be a maximum of condensation near the centre of the disc, because the particles there, not being in the direct course of the stream, will move but slowly, and consequently by reason of the equation

$$2a^2 \text{ hyp. log. } \frac{p}{\Pi} = -\omega^2,$$

p will be nearly Π: next there will be a sudden rarefaction on account of the confluence of the stream, as it enters between the disc and the plane surface: afterwards, the theory shews, there would be a continual increase of density up to the edge of the disc, were it not for some cause, very probably friction,

which occasions another maximum of density, the position of which depends on the energy of this cause.

(2) When air, subject to a given pressure Π, is made to issue through a small orifice into the atmosphere, the equation,

$$2a^2 \text{ hyp. log. } \frac{p}{\Pi} = \omega_1^2 - \omega^2,$$

applicable to this case, shews that the points of equal velocity are also points of equal pressure: consequently, that as the pressure at the surface of the stream must be nearly the atmospheric pressure, if the stream contract like water* there will be a converging surface at every point of which the velocity is nearly the same, and greater than the velocity at all points within it, so that the pressure within the surface will be greater than the pressure of the atmosphere. If a tube be fitted to the orifice, exactly of the shape of this surface, the pressure on every point of it will be that of the atmosphere: if the tube have more convergence than corresponds to the natural contraction of the stream, it will be pressed from within to without. Conceive the stream to pass between two plane elastic laminæ converging towards each other, and let their inclination be greater than that which the nature of the stream requires. Their extreme edges will then be pressed outward. By this means a small portion of the stream will become divergent; and if $\omega_2 =$ the velocity where the pressure is P, the atmospheric pressure, it will be seen by the equation

$$2a^2 \text{ hyp. log. } \frac{p}{P} = \omega_2^2 - \omega^2,$$

* The foregoing example makes this probable. See also Dr. Young's Lectures on *Natural Philosophy*, Vol. II. p. 534.

that as ω just before the stream enters the atmosphere must be greater than ω_2, p will be less than P, and the pressure will be from without to within. Thus the extreme edge of each lamina will have a tendency to vibrate simultaneously about two points; one vibration depends on its elasticity, the other on the mode of action of the fluid. The result will be a continual vibration about another fixed point. It is to be observed that the perfect elasticity of the lamina is not a necessary condition, and that the action of the fluid is such, that the vibrations would be kept up without intermission, if the vibrating substance were of a yielding nature like leather. It is, I think, by reason of a mode of action of the air, at least similar to that I have attempted to describe, that the general principle stated by Professor Airy (*Cam. Phil. Trans.* Vol. III. Part II. p. 370.) becomes applicable to Mr. Willis's very interesting experiment, illustrative of the mechanism of the vocal organs.

J. CHALLIS.

TRINITY COLLEGE,
Feb. 20, 1830.

XIX. *On Crystals found in Slags.*

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[Read *March* 22, 1830.]

THE crystals described in the following paper occur in slags found at the bottom of the furnaces in the iron works at Merthyr Tydvil, and at Birmingham. Their primary form is a right rhombic prism, of which Figs. 1 and 2 represent the usual modifications. The angles, which were measured by the reflective goniometer, approximate closely to those of olivine or peridot; their agreement in particular with the angles of peridot, according to Mitscherlich, who also used the reflective goniometer in measuring them, is almost perfect.

The numbers in the first column indicate the law of derivation of the planes, according to the method proposed by Professor Whewell, in the *Philosophical Transactions* for 1825.

		Slag.	Peridot, Mitscherlich.	Crysolite, Nauman.	Olivine, Phillips.
(1; 1; 0)	*MM'*	130,30	130,26	130,2	130,
(3; 2; 0)	*qq'*	110,40	....	....	....
(2; 1; 0)	*rr'*	94,40	94,34	94,3	94,16
(3; 1; 0)	*ss'*	72,	....	71,10	....
(5; 1; 0)	*tt'*	48,	....	....	....
(1; 0; 1)	*cc'*	81,38	81,17	80,53	80,
(1; 0; 2)	*dd'*	119,20	....	119,12	....
(0; 1; 2)	*aa'*	75,20	....	76,54	....
(3; 1; 2)	*cx*	146,	....	....	....

The planes *M*, are striated parallel to their common intersection. *q*, *t*, *x*, have not hitherto been observed in the natural minerals. *s*, *t* are too imperfect to be measured accurately.

Some of the crystals from Merthyr, exhibiting the secondary planes represented in Fig. 2, are transparent, of a dark olive green colour, and possess a high refractive power, the minimum deviations of yellow rays through *MM''* being 49°,32′ and 52°,48′. Opaque crystals from the same place having a semi-metallic lusture, and apparently containing a mechanical mixture of foreign matter, invariably take the form of Fig. 1. The crystals from Birmingham, which also have the form of Fig. 1, are black, and cleave readily in a direction perpendicular to the planes *M*, *M'*.

In chemical composition, peridot is found to be a silicate of magnesia and of protoxide of iron, that is, a compound of the magnesia and the protoxide of iron with silica, so that the oxygen of the two first is equal to that of the latter. It is worthy of notice, that Professor Mitscherlich has already found

minerals, having the form of peridot in the slags of the furnaces of Sweden and Germany, (Brewster's *Journal,* Vol. II. p. 130, and *Annales de Chèmie et de Physique,* Tom. XXIV.) Some of the varieties, of which he gives the analysis, consist almost entirely of protoxide of iron and silica, in the proportion of 36 of the former, to 16 of the latter; in others he found lime or magnesia, replacing part of the iron, so as always to make up a quantity of oxygen equal to that of the silica. These chemical constitutions may be expressed by the formulæ

$$Fe + S,\ \begin{matrix}Fe\\C\end{matrix} + S,\ \begin{matrix}Fe\\M\end{matrix} + S,$$

and all these cases are included in the chemical formula

$$\begin{matrix}Fe\\M\\C\end{matrix} + S \text{ where } Fe,\ M,\ C, \text{ \&c.}$$

are isomorphous or plesiomorphous; that is, may replace each other with no change of form and angle, or with a slight change of angle, and a preservation of the system of crystallization.

The mineral called Lievrite or Yenite has for its formula

$$4\,(Fe + S) + (C + S).$$

This is manifestly a particular case of the formula already given. Also $Zc + S$, the formula expressing the composition of silicate of zinc, is analogous to it, Zc being isomorphous with Fe. We might expect therefore the angles in these two cases to approximate to those of peridot. It will however be more convenient to compare them with the angles of the slag, than with those of peridot, the former having a greater number of secondary planes.

In cases where any of the planes are wanting, it will be necessary to complete the series, and to compute the angles between the interpolated planes.

The results of the comparison are presented in the following Table.

		Slag. Fig. 3.	Yenite. Fig. 4.	Silicate of Zinc. Fig. 5.
(1; 1; 0)	*MM'*	130,30	131,10*	133,44*
(3; 2; 0)	*qq'*	110,40	111,30	114,42*
(3; 1; 0)	*ss'*	72,	73,6	75,55
(3; 0; 2)	*uu'*	59,52*	57,40*	57,30
(1; 0; 1)	*cc'*	81,38	....	....
(3; 0; 4)	*vv'*	....	95,30	....
(1; 0; 2)	*dd'*	119,20	117,36*	117,20
(0; 3; 4)	*ww'*	....	53,10	....
(0; 1; 2)	*aa'*	75,20	74,8*	70,
(0; 1; 4)	*yy'*	....	113,	....
(0; 1; 6)	*zz'*	132,42*	132,24*	128,50

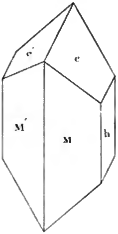
The computed angles are distinguished by an asterisk.

W. H. MILLER.

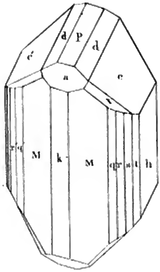
ST. JOHN'S COLLEGE,
March 22, 1830.

Slag

1



2



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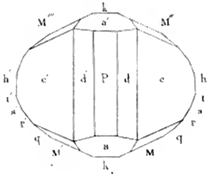
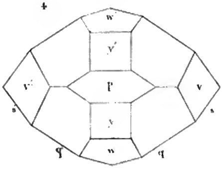


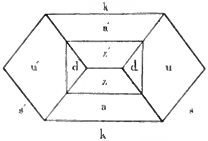
Yenite

4



Silicate of Zinc

5



XX. *On the Improvement of the Microscope.*

By H. CODDINGTON, M.A. F.R.S.

FELLOW OF TRINITY COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *March* 22, 1830.]

Among the numerous excellent suggestions which Dr. Brewster has from time to time thrown out to those engaged in the theory or the practice of Optics, there is one which appears to have been most unworthily, and most unaccountably, neglected. It is that of substituting a *sphere* for a *lens* in the construction of a microscope. This is the more surprising, as many persons of great eminence have, of late years, turned their attention to the improvement of this instrument, in which pursuit they have spared neither time, labor, nor expense. The only reason which I can give for this is, that as, until the investigations of Professor Airy, which are contained in the present volume of the Transactions of this Society, nobody, with the exceptions of Dr. Young and Dr. Wollaston, ever dared to approach the thorny subject of the oblique refraction* of a pencil of rays by a lens, almost

* I allude here to the refraction of light, considered as homogeneous. The *chromatic dispersion* introduces another source of error, which has been very successfully overcome. I have found the *colour* most completely *corrected* in microscopes which were, in other respects, so ill constructed, as to be nearly useless.

all other persons have been satisfied with endeavouring to show, as distinctly as possible, one individual point of an object, trusting that the rest would follow of itself, or giving up, as hopeless, the idea of producing a good and large field of view.

To those who have studied the construction of the compound microscope, an analogy presents itself, very naturally, between that instrument and the telescope. In each there is an image formed, which is seen through one or more lenses, constituting what is technically termed in the former case, the ***body***, in the latter the *eye-piece*.

The progress of these instruments has been curiously similar in some respects. The first step of any consequence in the case of the telescope, was Huyghens's eye-piece, which, besides the merit supposed by its author, of diminishing the errors arising from aberration, had one, much more important, which he did not contemplate, the correction of the coloured fringes, seen about every point of the image, except that precisely in the centre*. Ramsden then succeeded in making an eye-piece, which gives a flat field of view, when that point is particularly important, and finally, the instrument has been made perfect, by substituting for the simple object glass, an achromatic and aplanatic combination of lenses. In the compound microscope, the first point, (the correction of the coloured fringes,) has been completely attained; on the second, much labour has been bestowed by practical opticians, but with little success; the third has lately occupied some of the most distinguished theorists and artisans, who have been eminently successful, but the

* This is the more remarkable, as this point was completely attained at once, whereas the other was long but half gained. It is but of late years, that even an approximation to the proper forms of the lenses has been made in the achromatic eye-piece.

difficulty and the expense necessarily attending their processes, are so great, that but few persons can derive any benefit from their exertions*.

In making a comparison between the telescope and microscope, it must be observed, that some difficulties, and sources of error, which in the former are so small as to have been overlooked, are in the latter of the greatest and most palpable importance. The image produced by the object glass of a telescope is usually considered as perfectly plane, and equally distinct in all its parts, and this supposition is quite sufficiently accurate, because although the image given by a lens with centrical pencils, is on the whole very much curved and very indistinct, so small a part of it is employed in this case, and that only the most perfect, that the defects are usually quite insensible in practice.

I have shown†, after Dr. Young and Professor Airy, that if we represent by

λ, the aperture of the object-glass,

z, the distance of a point of the image from the axis,

f, the focal length of the lens,

k, the distance of the image from the lens,

the indistinctness is proportional to the diameter of the least space over which a pencil is diffused, the value of which is

$$\frac{\lambda z^2}{2kf}.$$

* Mr. Tulley has just finished an achromatic microscope ordered for Lord Ashley, about six months ago. This instrument, which I have seen, is a masterpiece of art, but I believe that the above eminent optician has been obliged to make the object-glass with his own hands, and the price is far beyond the reach of most naturalists.

† Treatise on the Reflexion and Refraction of Light, Art. 145.

Now in a telescope, z being nearly equal to the semi-aperture of the field glass, is very much less than f, to which k is equal, and as a high magnifying power is produced by means of a powerful eye-piece, applied to an object-glass which is never changed, and that the apertures of the lenses used for eye-pieces, of the same kind, are usually proportional to their focal lengths, the higher the magnifying power, the less is the fraction

$$\frac{z^2}{kf} \text{ or } \frac{z^2}{f^2}.$$

For instance, in a 5 foot telescope, it is seldom, if ever, greater than $\frac{1}{14400}$, and often very much less, so that the value of the quantity

$$\frac{\lambda z^2}{2f^2} \text{ is about } \frac{1}{7200}.$$

In a microscope, on the other hand, f is a very small quantity, though k is not so, and the magnifying power is raised by applying an object-glass of shorter focus to the same body. The following values are, I believe, such as might fairly occur:

$$\lambda = \frac{1}{20},$$

$$z = \frac{1}{2},$$

$$f = \frac{1}{4},$$

$$k = 3.$$

These give $\frac{\lambda z^2}{2kf} = \frac{1}{60}$, which as, with different object-glasses, z and k are constant, and λ usually proportional to f, may be considered as its general value.

Again, in a telescope, the portion of the image used is sensibly flat, though the radius of curvature of every such image, is about $\frac{3}{8}$ths of the focal length of the object-glass*: but in the microscope it is evidently far otherwise, so that were the whole image distinct, it would still be impossible to have any great extent of it distinctly visible at once; and this objection applies in full force to the most perfect achromatic object-glass. Now with a sphere, properly cut away at the center so as to reduce the aberration, and dispersion, to insensible quantities, which may be done most completely† and most easily, as I have found in practice, the whole image is perfectly distinct, whatever extent of it be taken, and the radius of curvature of it is no less than the focal length, so that the one difficulty is entirely removed, and the other at least diminished to one half.

Besides all this, another advantage appears in practice to attend this construction, which I did not anticipate, and for which I cannot now at all account. I have stated‡ that when a pencil of rays is admitted into the eye, which, having passed without deviation through a lens, is bent by the eye, the vision is never free from the coloured fringes produced by excentrical dispersion. Now with the sphere I certainly do not perceive this defect, and I therefore conceive that if it were possible to make the spherical glass on a very minute scale, it would be the most perfect simple microscope, except perhaps Dr. Wollaston's

* I suppose the lens to be of *glass*. In any case it is less than half the focal length.

† The only limit in practice to the diminution of the aperture, is the danger of the glass breaking, for the loss of *light* is trifling, in comparison of that which is unavoidable in many other constructions.

‡ Treatise on the Eye and Optical Instruments, Art. 325.

doublet, than which I can hardly imagine anything more excellent as far as its use extends, its only defects being the very small field of view, and the impracticability of applying it, except to transparent objects, seen by transmitted light.

Now the sphere has this advantage, that whereas it makes a very good simple microscope, it is more peculiarly fitted for the object-glass of a compound instrument, since it gives a perfectly distinct image of any required extent, and that, when combined with a proper eye-piece, it may without difficulty be employed for opaque objects. I have therefore endeavoured so to combine it, and this has been my principal difficulty; for the systems of lenses which I have found employed for this purpose, are so improperly constructed, that I have been forced to have one made from original calculations, and get tools constructed on purpose, which has necessarily been attended with some delay.

The principle which I have adopted, after one or two previous trials, may be explained as follows.

One great cause of the excellency of Huyghens's eye-piece, is the condition which he himself designed to fulfil, namely, that the bending of the pencil is equally divided between the two lenses. Now this may be done for a microscope, thus:

Let O (Fig. 1.) be the center of the object-glass,
F the place of the field-glass,
E .... eye-glass.
Let $OF = 2$ inches (for example)
$FE = 1$ inch.
And let the focal length of the field-glass be 1 inch.
.... eye-glass ... $\frac{1}{2}$ inch.

These values satisfy the conditions of achromatism, and it will easily be seen, that if Y be the place where the pencil tends to

cross the axis after refraction at the field-glass, and z that where it actually crosses after emerging from the eye-glass, the angle of flexure, at each lens, is double of the original inclination of the pencil to the axis.

This simple system is however not applicable, as it is impossible to satisfy the condition necessary for perfect distinctness, much less that for destroying, as far as possible, the convexity of the field. These may however be very readily satisfied by employing two lenses of equal power, in each place, instead of one. The most proper forms of the lenses are those shown in Fig. 2, the field-glasses and the second eye-glass being of the meniscus form, and the first eye-glass equi-convex. I have found no sensible error arise from the substitution of plano-convex lenses for the meniscus glasses, which are difficult and expensive to form. Theory indicated a further flattening of the field, to be made by separating the eye-glasses a little, which requires the distance of the first eye-glass from the field-glasses, to be diminished by about half as much*; I cannot say however that I perceive any improvement arising from this alteration in practice, and as the field is quite flat enough with the eye-glasses in contact, and any further diminution of the apparent convexity, can be gained only by a sacrifice of distinctness, I cannot on the whole recommend it. I have not however yet had the instrument in a sufficiently perfect state of adjustment, in other respects, to be able to give a decided opinion on this point. This system, as it will easily be seen, gives a magnifying power of 3 to the eye-piece, so as to multiply, by that number, the power of the object-glass. It would be easy, if necessary, to produce a higher magnifying power, by employing lenses of

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* The first eye-glass should in this case, be a little less curved on the lower surface and a little more on the upper, but it is hardly worth while to alter the form in practice.

shorter focal lengths, regard being had, in each case, to the proper condition of achromatism. Thus several different eye-pieces might be inserted, at pleasure, into one tube, in the same manner as it is usual to vary the magnifying power of a telescope*. I have not yet tried the effect of this, but I suppose it may be necessary in applying the microscope to opaque objects, as the difficulty of illuminating them almost precludes the use of a powerful object-glass.

I do not pretend to give this as a perfect instrument—much less as one that will answer all purposes; but having tried it, in a very rough state, and with a moderate magnifying power, on various delicate test objects, all of which it shows very satisfactorily†, not excepting the striæ on the scales of the Podura, which Mr. Pritchard, the inventor of the diamond and sapphire lenses, says are only just discernible with the most perfect instruments, I see no reason to doubt that, when carefully executed, it will be found very effective, and that the naturalist may be furnished, at an expense not exceeding five or six guineas, with a microscope which will perform nearly all that can be expected from that instrument.

Fig. 3. represents the microscope, as I have directed it to be made by Mr. Cary.

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* For example, if two equal plano-convex lenses having a joint focal length of half an inch, be placed at a distance of two inches and a half from the centre of the object-glass, and the eye-glasses be like those of the former eye-piece, but of focal lengths half an inch and one-third of an inch, the distance between the two pairs being $\frac{7}{18}$in. the power of the object-glass will be multiplied about eight times.

† Mr. R. Brown's active molecules may be very pleasantly observed with this microscope, with a power of about 360.

H. CODDINGTON.

TRINITY COLLEGE,
April 23, 1830.

Fig. 1.

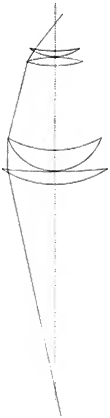
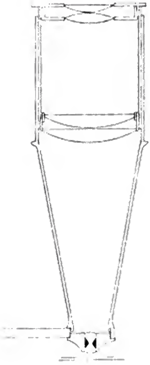


Fig. 2.

Fig. 3.



XXI. *On the General Properties of Definite Integrals.*

By R. MURPHY, B.A.

FELLOW OF CAIUS COLLEGE, AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *May* 24, 1830.]

THE modern physical theories, particularly those on the propagation of heat, and the distribution of electricity, attach a new importance to the subject of definite integrals; being used in the former case as the simplest means of integrating the partial equations which express the variations of temperature in bodies, and in the latter case entering implicitly the equations for the equilibrium and motion of electricity developed on the surfaces of bodies — in many problems of this nature the form of the function under the sign of definite integration is unknown, and the resolution of equations in which terms of this nature are found, is generally attended with very considerable difficulty — it is obvious that considerable advantages might in these and other researches be expected to result from the study of the *general* properties of definite integrals, that is, such properties as hold true whatever be the function under the sign of integration.

For convenience sake, the definite integral of $f(x)$, taken with respect to x between stated limits, is denoted by $D.f(x)$; in the succeeding theorem the limits are

$$\left.\begin{array}{l} x = a + h\sqrt{-1} \\ x = a - h\sqrt{-1} \end{array}\right\},$$

which are capable of representing any whatever.

THEOREM I.

$$\text{Put} \quad u = \frac{\epsilon^{\frac{\overline{n+1}.\pi}{h}.a}}{1.2.3...n} \times \frac{d^n\left\{f(a).\epsilon^{-\frac{\pi}{h}.a}\right\}}{d\left(\epsilon^{\frac{\pi}{h}.a}\right)^n}.$$

$$\text{And} \quad u_1 = \frac{\epsilon^{-\frac{\overline{n+1}.\pi}{h}.a}}{1.2.3....n} \times \frac{d^n\left\{u.\epsilon^{\frac{\pi}{h}.a}\right\}}{d\left(\epsilon^{-\frac{\pi}{h}.a}\right)^n};$$

then shall

$$D.f(x) = 2h\sqrt{-1}.u_1,$$

n being made infinite in the value of u_1.

To prove this, let $\phi(x)$ be the indefinite integral of $f(x)$,

$$\text{and let } \frac{d\phi(a)}{da} = \phi'(a), \quad \frac{d^2\phi(a)}{da^2} = \phi''(a), \text{ \&c.}$$

Suppose $c_1, c_2...c_n...c_{2n}$ any series of quantities $2n$ in number, and $a_1, a_2....a_{2n}$ formed from these by the following law:

$$a_1 = \phi'(a) - hc_1\frac{d.\phi'(a)}{da},$$

$$a_2 = a_1 - hc_2.\frac{da_1}{da},$$

$$a_3 = a_2 - hc_3.\frac{da_2}{da},$$

$$\text{.........................}$$

$$a_{2n} = a_{2n-1} - hc_{2n}.\frac{da_{2n-1}}{da},$$

if we actually make these successive substitutions we shall get a_{2n} in terms of $\phi'(a)$ and its differential coefficients, and the law of the series is extremely simple;

for if S_1 represent the sum of $c_1, c_2 \dots . c_{2n}$,
S_2 that of their products two by two,
&c.

then we shall have

$$a_{2n} = \phi'(a) - hS_1 . \phi''(a) + h^2 . S_2 . \phi'''(a) - \&c.$$

Suppose now that $\quad c_1 = \frac{1}{\pi}, \quad c_2 = \frac{1}{2\pi} \dots\dots c_n = \frac{1}{n\pi}$,

$$c_{n+1} = -\frac{1}{\pi}, \quad c_{n+2} = -\frac{1}{2\pi} \dots\dots c_{2n} = -\frac{1}{n\pi},$$

then it is plain that when n is infinite, S_1 = sum of the reciprocals of the roots of the equation $\frac{\sin(x)}{x} = 0$,

S_2 = sum of the products of these reciprocals two by two,
&c.

that is, $\quad S_1 = 0, \quad S_2 = -\frac{1}{1.2.3}, \quad S_3 = 0, \quad S_4 = \frac{1}{1.2.3.4.5}$, &c.

$$\therefore a_{2n} = \phi'(a) - \frac{h^2}{1.2.3} . \phi'''(a) + \frac{h^4}{1.2.3.4.5} . \phi^{v}(a) - \&c. \text{ when } n = \infty$$

$$= \frac{\phi(a + h\sqrt{-1}) - \phi(a - h\sqrt{-1})}{2h\sqrt{-1}}, \text{ by Taylor's theorem,}$$

$$= \frac{D . f(x)}{2h\sqrt{-1}}, \text{ evidently} \dots\dots\dots\dots\dots\dots\dots (1).$$

But the value of a_{2n} may be found in another manner, thus:

$$a_1 = \phi'(a) - \frac{h}{\pi} . \frac{d . \phi'(a)}{da} = -\frac{h}{\pi} . \epsilon^{\frac{\pi}{h} . a} . \frac{d\left(\phi'(a)\epsilon^{-\frac{\pi}{h} . a}\right)}{da} = -\epsilon^{\frac{2\pi}{h} . a} \frac{d\left(\phi'(a)\epsilon^{-\frac{\pi}{h} . a}\right)}{d\left(\epsilon^{\frac{\pi}{h} . a}\right)} .$$

Similarly,

$$a = -\frac{h}{2\pi}\cdot\epsilon^{\frac{2\pi}{h}\cdot a}\,\frac{d\left(x_1\cdot\epsilon^{-\frac{2\pi}{h}\cdot a}\right)}{da} = -\tfrac{1}{2}\cdot\epsilon^{\frac{3\pi}{h}\cdot a}\,\frac{d\left(a_1\epsilon^{-\frac{2\pi}{h}\cdot a}\right)}{d\left(\epsilon^{\frac{\pi}{h}\cdot a}\right)} = \tfrac{1}{2}\cdot\epsilon^{\frac{3\pi}{h}\cdot a}\,\frac{d^2\left(\phi' a\cdot\epsilon^{-\frac{\pi}{h}\cdot a}\right)}{d\left(\epsilon^{\frac{\pi}{h}\cdot a}\right)^2}$$

....

$$a_n = \frac{(-1)^n}{1.2....n}\cdot\epsilon^{\overline{n+1}\cdot\frac{\pi}{h}\cdot a}\cdot\frac{d^n\left(\phi'(a)\cdot\epsilon^{-\frac{\pi}{h}\cdot a}\right)}{d\left(\epsilon^{\frac{\pi}{h}\cdot a}\right)^n}.$$

Comparing this with the value of u, we have

$$a_n = (-1)^n\cdot u.$$

Exactly a similar process applied to the remaining quantities $a_{n+1},\ a_{n+2}\ldots.a_{2n}$, gives

$$a_{2n} = \frac{(-1)^n}{1.2....n}\cdot\epsilon^{-\overline{n+1}\cdot\frac{\pi}{h}\cdot a}\,\frac{d^n\left(a_n\cdot\epsilon^{\frac{\pi}{h}\cdot a}\right)}{d\left(\epsilon^{-\frac{\pi}{h}\cdot a}\right)^n},$$

that is, $a_{2n} = \dfrac{\epsilon^{-\overline{n+1}\cdot\frac{\pi}{h}\cdot a}}{1.2.....n}\cdot\dfrac{d^n\left(u\cdot\epsilon^{\frac{\pi}{h}\cdot a}\right)}{\left(d\cdot\epsilon^{-\frac{\pi}{h}\cdot a}\right)^n}$,

or $a_{2n} = u_1$ .... (2).

Equate the values of a_{2n} in the equations (1) and (2), and we get

$$D.f(x) = 2h\sqrt{-1}\cdot u_1.$$

THEOREM II. Supposing the limits of the integral now to be $x = 0$, $x = \infty$, then shall

$$D.f(a.x^x) = f(a) - \frac{1}{2^2}\cdot\frac{d.f(a)}{d(\log.a)} + \frac{1}{3^3}\cdot\frac{d^2.f(a)}{d(\log.a)^2} - \&c.$$

For let $f(a) = \phi(\log.a)$;

therefore $f(ax^x) = \phi(\log.a + x\log.x)$

$$= \phi(\log.a) + x\log.x.\,\phi'(\log.a) + \frac{(x\log.x)^2}{1.2}.\,\phi''(\log.a) + \&c.$$

if we now integrate each term between the given limits, and restore for $\phi(\log. a)$ its value $f(a)$, we shall get the above series.

In the next theorem the limits of the integral are supposed to be a, $a+2h$, which may represent any whatever; the series which expresses the value of the definite integral of any function is not only remarkable for the simplicity of its form, but likewise because it gives the law of the errors made by taking any number of its first terms for the definite integral.

THEOREM III. Let $\phi_n(a)$ denote the error made by taking the first $n-1$ terms of the following series for the true value of the definite integral of $\phi(x)$; then shall

$$\begin{aligned} D.\phi(x) = 2h\phi(a) + \frac{h}{1}.\frac{d.\phi_1(a)}{da} - \frac{h}{3}.\frac{d.\phi_2(a)}{da} + \frac{h}{3}.\frac{d.\phi_3(a)}{da} \\ - \frac{h}{5}.\frac{d.\phi_4(a)}{da} + \frac{h}{5}.\frac{d.\phi_5(a)}{da} \\ - \&c. \end{aligned}$$

Thus if $\phi(x)$ be the ordinate of a curve, $D.\phi(x)$ is the area contained between the curve, the two extreme ordinates, and the axis of x, the first term represents only the inscribed parallelogram, the two first represent the trapezoid formed by the chord, the extreme ordinates, and the axis of x; the three first terms represent the parabolic segment, which has its extreme points common with the curve, which was the approximation used by Simpson, and the remaining terms give always the nearest approximations (when h is small) that may be made to the true area. For if we suppose $f(x)$ to be the indefinite integral, and u to be the true value of the definite integral, we must have

$$f(a+2h) - f(a) = u,$$

and if we expand the first term by Taylor's theorem, we shall get a linear equation of infinite dimensions, which being resolved

in the ordinary method by putting ϵ^{ma} for $f(a)$, the equation for determining the values of m is $\epsilon^{2mh} - 1 = 0$; now this reducing equation which is obtained only from the left side of the linear equation, can never be the same with respect to two linear equations, unless their left sides be identical. But in this case, since $\phi(a) = \dfrac{df(a)}{da}$, if we put

$$U = 2h\frac{d.f(a)}{da} + \frac{h}{1}.\frac{d.f_1(a)}{da} - \frac{h}{3}.\frac{d.f_2(a)}{da}, \text{ \&c.}$$

$f_n(a)$ being used in a similar sense to $\phi_n(a)$, the resulting equation on the supposition of $U=u$, will be exactly identical with $\epsilon^{2mh} - 1 = 0$, as is obvious from the known equation

$$\epsilon^{2mh} = 1 + \cfrac{2mh}{1 - \cfrac{mh}{1 + \cfrac{\frac{1}{3}mh}{1 - \cfrac{\frac{1}{3}mh}{1 + \text{\&c.}}}}}$$

THEOREM IV. If $a, a+h$, be the limiting values of x, and α any quantity whatever, then shall

$$D.\phi(x) = h\left\{\phi(a+\alpha) + \frac{\overline{h-2\alpha}}{1.2}.\phi'(a+2\alpha) + \frac{\overline{h-3\alpha}^2}{1.2.3}.\phi''(a+3\alpha) + \frac{\overline{h-4\alpha}^3}{1.2.3.4}.\phi'''(a+4\alpha), \text{ \&c.}\right\},$$

$\phi'(a)$, $\phi''(a)$, &c. being the differential coefficients of $\phi(a)$.

For the terms in the preceding series which contain

$$\phi^{(n+m)}(a).\alpha^n,$$

when respectively expanded by Taylor's theorem, are those after the m^{th} place, and if the coefficient of that quantity be collected out of these different expansions, its value is

$$\frac{h^m}{1.2...m \times 1.2...n}\left\{(m+1)^{n-1} - n(m+2)^{n-1} + \frac{n.\overline{n-1}}{1.2}(m+3)^{n-1}, \text{ \&c.}\right\},$$

which for any positive and integer value of n is always 0; and therefore the given series is simply equivalent to

$$h\phi(a) + \frac{h^2}{1.2} \cdot \phi'(a) + \frac{h^3}{1.2.3} \cdot \phi''(a), \text{ \&c.}$$

which is evidently equal to the definite integral of $\phi(x)$.

Most of these theorems may, without considerably altering their forms, be applied to the definite integrals in finite differences, but it would too much extend this paper to insert them with their demonstrations: the following, for instance, is analogous to Theorem III. D' being used to denote a definite integral in finite differences, a and $a+mh$ the limiting values of x, h being the increment of x or a, then we shall have

THEOREM V.

$$D'.\phi(x) = m.\phi(a) + \frac{m-1}{1} \cdot \frac{\Delta.\phi_1(a)}{2} - \frac{m+1}{3} \cdot \frac{\Delta.\phi_2(a)}{2} + \frac{m-2}{3} \cdot \frac{\Delta.\phi_3(a)}{2} - \frac{m+2}{5} \cdot \frac{\Delta.\phi_4(a)}{2} + \text{\&c.}$$

where $\phi_n(a)$ denotes the error made by taking $(n-1)$ terms of the series for the true definite integral.

From this theorem and any others of the same nature, in which each term depends on the error made by taking a certain number of the preceding terms for the whole, many remarkable theorems with respect to the reproduction of functions, even when not continuous, may be deduced: for example,

* If there be n rows of quantities, the first row entirely arbitrary, any even row, as the $2m^{\text{th}}$, formed so that the vertical difference

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* In series formed after this manner, it is a curious property that the difference of two terms in n^{th} row, which are $\frac{n+1}{2}$ places distant $= \frac{n+1}{2} \times$ difference of two consecutive terms in the row preceding.

of any term $= \frac{-m}{n-2m} \times \frac{1}{2}$ the difference taken horizontally, and in the odd rows (as the $\overline{2m+1}^{\text{th}}$) the law be the same, except that the multiplier is $\frac{n-m}{n-2m}$, then the n^{th} row will always be a reproduction of the first. Example for five rows:

$$\begin{array}{llllll}
{}^{*}1, & 4, & 9, & 16, & 25, & \&c. \quad \text{First row.} \\
\frac{5}{2}, & \frac{13}{2}, & \frac{25}{2}, & \frac{41}{2}, & \frac{61}{2}, & \&c. \\
\frac{12}{6}, & \frac{34}{6}, & \frac{68}{6}, & \frac{114}{6}, & \frac{172}{6}, & \&c. \\
\frac{28}{6}, & \frac{58}{6}, & \frac{100}{6}, & \frac{154}{6}, & \frac{220}{6}, & \&c. \\
\frac{6}{6}, & \frac{24}{6}, & \frac{54}{6}, & \frac{96}{6}, & \frac{150}{6}, & \&c. \quad \text{Fifth row; same as first.}
\end{array}$$

The inverse problem, viz. given the definite integral of a function in different circumstances, to find the form of the function itself, is susceptible of various solutions, as those circumstances are themselves varied: thus, if D_1 be the definite integral of any function of x from $x = a - h\sqrt{-1}$ to $x = a + h\sqrt{-1}$,

$$\begin{array}{llll}
D_3 & \text{from } x = a - 3h\sqrt{-1} & \text{to} & x = a + 3h\sqrt{-1}, \\
D_5 & \text{from } x = a - 5h\sqrt{-1} & \text{to} & a + 5h\sqrt{-1}, \\
\&c. & & &
\end{array}$$

then if we take the sum of the series

$$S = D_1 - \frac{D_3}{9} + \frac{D_5}{25}, \quad \&c.$$

* Thus if Δ_1 be the vertical, and Δ the horizontal differences, and u_1 any term in the first line, u_2 the corresponding in the second, &c.

$$\text{then } \Delta_1(u_1) = \frac{n}{n} \cdot \frac{\Delta . u_1}{2}, \qquad \Delta_1(u_2) = -\frac{1}{n-2} \cdot \frac{\Delta . u_1}{2},$$

$$\Delta_1(u_3) = \frac{n-1}{n-2} \cdot \frac{\Delta . u_2}{2}, \quad \Delta_1(u_4) = \frac{-2}{n-4} \cdot \frac{\Delta . u_3}{2}.$$

&c. &c.

and change a into x, we shall have the function itself by dividing $2S$ by $\pi h\sqrt{-1}$, but when the above series is divergent, it becomes troublesome to calculate the *Analytical* values of the divergent parts, to remove which difficulty we should attend to the values of the divergent series arising from differentiating successively the equation

$$\frac{\pi}{4} = \cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta, \text{ \&c.}$$

thus 0 should be substituted for $1 - 3 + 5 - 7$, &c.

0 for $1^3 - 3^3 + 5^3 - 7^3$, &c.

&c.

Definite integrals may be applied to the expansion of functions when they contain negative powers of x, and they serve to determine the coefficients of such terms.

Thus, if we wish to find P, the coefficient of x^p in $f(x)$, when such a term enters, we have

$$\frac{f(x)}{x^p} = \left.\begin{matrix} P + Qx^{\alpha} + Rx^{\beta} + \text{\&c.} \\ + qx^{-a} + rx^{-b} + \text{\&c.} \end{matrix}\right\},$$

the upper line containing the positive, and the lower the negative, powers.

Multiply both sides by $\frac{dx}{x}$, and integrate from $x = -1$ to $x = +1$. Hence,

$$D.\frac{f(x)}{x^{p+1}} = -P\log.(-1) + \left.\begin{matrix} \frac{2Q}{\alpha} + \frac{2R}{\beta} + \text{\&c.} \\ -\frac{2q}{a} - \frac{2r}{b} - \text{\&c.} \end{matrix}\right\},$$

supposing α, β, a, b, &c. to be odd, for the terms containing even powers entirely disappear in the integration.

$$\text{Again,}\quad x^p f\left(\frac{1}{x}\right) = \left.\begin{array}{l} P + qx^a + rx^b, \ \&c. \\ \quad + Qx^{-\alpha} + Rx^{-\beta}, \&c. \end{array}\right\};$$

$$\text{and therefore}\quad D\left\{x^{p-1} f\left(\frac{1}{x}\right)\right\} = -P \log.(-1) + \frac{2q}{a} + \frac{2r}{b}, \ \&c. \\ -\frac{2Q}{\alpha} + \frac{2R}{\beta}, \ \&c.$$

$$\text{whence}\quad P = \frac{1}{2\pi\sqrt{-1}} D\left\{\frac{f(x)}{x^{p+1}} + x^{p-1}.f\left(\frac{1}{x}\right)\right\}.$$

Hitherto we have taken the limits of the integral independent of x, the same results hold in general when x is used for a; but it is to be observed that when the limits contain x, (as $x-h$, $x+h$,) the integral remains a function of x, and therefore is capable of being integrated again; and the result may be called the second definite integral of the given function, which being integrated between the same limits, will give the third definite integral, and so on: we shall denote by D', D'', D''', &c. the successive definite integrals taken in this point of view: In the following theorem which is similar to Theorem I, the limits are $x-h\sqrt{-1}$ and $x+h\sqrt{-1}$; it comprises that theorem as a particular case, and admits of a similar proof.

THEOREM VI.

$$\text{Put}\quad u = \frac{\epsilon^{\frac{\overline{n+1}.\pi}{h}.x}}{(1.2.3.....n)^m} \times \frac{d^{m.n}\left(f(x).\epsilon^{-\frac{\pi}{h}.x}\right)}{d\left(\epsilon^{\frac{\pi}{mh}.x}\right)^{mn}},$$

$$\text{and}\quad u_1 = \frac{\epsilon^{-\frac{\overline{n+1}.\pi}{h}.x}}{(1.2.3.....n)^m} \times \frac{d^{mn}\left(u.\epsilon^{\frac{\pi}{h}.x}\right)}{d\left(\epsilon^{-\frac{\pi}{mh}.x}\right)^{mn}};$$

then shall $(2h\sqrt{-1})^m.u_1$ be the value of the m^{th} definite integral of $f(x)$; n being made infinite in the value of u_1.

This theorem by making $m = -1$ shews how to determine the quantity *of which* the definite integral is $f(x)$, the differential coefficients are in this case of a negative order $-n$; that is, they represent the n^{th} integrals of the quantities under them.

If we know the first, second, third, &c. definite integrals, we may find the function itself by the following formula.

Theorem VII. If $x - h$, $x + h$ be the limits of integration, then shall

$$2hf(x) = D'f(x) - \frac{1}{2} \cdot \frac{1}{3 \cdot 2^2} \frac{d^2 \cdot D'''f(x)}{dx^2}$$

$$+ \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 2^4} \cdot \frac{d^4 \cdot D^{v}f(x)}{dx^4}, \text{ \&c.}$$

these general properties of definite integrals might be much more extended, but the consideration of them would extend this paper to an inconvenient length; but we may observe that in studying these properties we should divide the functions under the sign of integration into large classes possessed of some common property, such as vanishing when $x = \infty$, &c.: the results do not here possess altogether the generality of the former, but they are more remarkable, and even more useful in analysis. To this part of the subject, however, I shall not at present further allude, but conclude with observing that I have subjoined as an illustration of the use of the preceding methods,—the general resolution of Riccati's Equation, by means of definite integrals.

Similarly, $$\int_h \frac{\phi\left(\frac{1}{h}\right) x \epsilon^{h x^{\frac{1}{a}}}}{h} = S \log.(-1) - 2T - \tfrac{2}{3}W - \&c.$$
$$+ 2t + \tfrac{2}{3}w + \&c.$$

whence $$\frac{x}{2}\int_h \frac{\phi(h)\,\epsilon^{\frac{x^{\frac{1}{a}}}{h}} + \phi\left(\frac{1}{h}\right)\epsilon^{h x^{\frac{1}{a}}}}{h} = S \log.(-1),$$

the limits of h being -1 and $+1$.

Again, if $$\phi'(h) = \epsilon^{h}.\, h^{a} \textstyle\int_h \epsilon^{-h} h^{-a-1},$$

the integral commencing when $h=0$, we have by the very same process,

$$\tfrac{1}{2}\int_h \frac{\phi'(h)\,\epsilon^{\frac{x^{\frac{1}{a}}}{h}} + \phi'\left(\frac{1}{h}\right).\,\epsilon^{h x^{\frac{1}{a}}}}{h} = S' \log.(-1);$$

if therefore we denote $cx\phi(h) + c'.\phi'(h)$ by $F(h)$, which function contains two arbitrary constants, we get

$$y = \int_h \frac{F(h).\,\epsilon^{\frac{x^{\frac{1}{a}}}{h}} + F\left(\frac{1}{h}\right).\,\epsilon^{h x^{\frac{1}{a}}}}{h},$$

from $h=-1$ to $h=+1$, such is the complete value of y; from hence the value of u, namely,

$$u = \frac{dy}{Aydt} = \frac{dy}{Aa.ydx},$$

is known, and u will lose none of its generality by making $C'=1$, for the constants evidently enter u only in the form $\frac{C}{C'}$, which may be replaced by C, without losing ought of its generality.

After the value of S had been found, we might have found the value of y, without staying to find S'; for since

$$\frac{d^2y}{dx^2} = \frac{1}{a^2} \cdot x^{\frac{1}{a}-2} \cdot y, \quad \text{and} \quad \frac{d^2S}{dx^2} = \frac{1}{a^2} \cdot x^{\frac{1}{a}-2} \cdot S;$$

$$\therefore\ S\frac{d^2y}{dx^2} - y\frac{d^2S}{dx^2} = 0; \qquad \therefore\ S\frac{dy}{dx} - y\frac{dS}{dx} = \text{const.} = \beta;$$

$$\therefore\ y = \beta S \int \frac{dx}{S^2},$$

$$\text{and} \qquad u = \frac{1}{Aa}\left\{\frac{dS}{Sdx} + \frac{1}{S^2 \int \frac{dx}{S^2}}\right\}.$$

R. MURPHY.

CAIUS COLLEGE,
May 24, 1830.

A

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PLATES.

**** The* PLATES *are wrong numbered, but may be distinguished by observing that some are marked with Arabic and some with Roman numerals.*

PLATE 1. refers to Mr. Airy's Paper on the Spherical Aberration of Eye-pieces, No. I.

2. refers to Mr. Airy's Paper on the Theory of Escapements, No. III.

I. refers to Mr. Willis's Paper on the Pressure of a Stream of Air upon a Plate, No. IV.

IV. refers to Mr. Morton's Paper on the Focus of a Conic Section, No. VIII.

4. refers to Mr. Willis's Paper on Vowel Sounds, No. X.

6. refers to Mr. Challis's Paper on the Vibrations of Elastic Fluids, No. XI.

7. refers to Mr. Miller's Paper on Crystals, No. XV.

8. ———————————————— No. XIX.

9. refers to Mr. Coddington's Paper on the Microscope.

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